Scalar-flat Kähler metrics with conformal Bianchi V symmetry

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Prim Plansangkate
prim.p@psu.ac.th
(Joint work with M. Dunajski (2011, Class. Quantum Grav.))

Prince of Songkla University, Thailand

Scalar-flat Kähler metrics with conformal Bianchi V symmetry

- A conformal structure $[g]$ on a Riemannian 4-manifold M
  - with anti-self-dual (ASD) Weyl curvature
  - admitting

    (i) a cohomogeneity-one (Bianchi V) metric g

    $$[X_0, X_1] = X_1, \quad [X_0, X_2] = X_2, \quad [X_1, X_2] = 0.$$

    (ii) a Kähler metric g_K (zero Ricci scalar)

Cohomogeneity-One Metrics

- $M = \mathbb{R} \times G$, where G is 3-dim isometry group.

$$g = dt^2 + h_{jk}(t)\, \lambda^j \odot \lambda^k, \qquad j, k = 0, 1, 2,$$

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- Diagonal ASD metrics: (Tod 1995)
  - Every ASD Bianchi V metric is conformally flat.
  - Question: Are there non-diagonal ASD Bianchi V Kähler metrics?

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- Non-diagonal ASD metrics:
  (Maszczyk-Mason-Woodhouse 1994, Maszczyk 1996)

Construction

Let $M = \mathbb{R} \times G, \quad g = V^{00'} \odot V^{11'} - V^{01'} \odot V^{10'}$

- $V_{00'} = \frac{\partial}{\partial t} + i\frac{R}{t}, \quad V_{11'} = \frac{\partial}{\partial t} - i\frac{R}{t}, \quad V_{01'} = P, \quad V_{10'} = -\overline{P}$

  $P, \overline{P}, R$ tangent to G & left-invariant, $\frac{\partial}{\partial t}$ tangent to $\mathbb{R}$,

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- ASD condition: (Penrose 1976, Mason-Woodhouse 1996)

  There exists two functions f_0, f_1 on $M \times \mathbb{CP}^1$ such that the distribution

  $$l = V_{00'} - \lambda V_{01'} + f_0 \frac{\partial}{\partial \lambda}, \quad m = V_{10'} - \lambda V_{11'} + f_1 \frac{\partial}{\partial \lambda}$$

  is integrable, i.e. $[l, m] = 0$ modulo l, m.

- Kähler condition: (Pontocorvo 1992, Hitchin 1995)
  $\lambda f_0 + f_1$ (a quartic in λ) has two distinct zeros of order 2.

- Cf. $$L = \frac{\lambda l + m}{\lambda f_0 + f_1}, \qquad M = \frac{f_1 l - f_0 m}{\lambda f_0 + f_1}$$

  Set $\lambda f_0 + f_1 = 2t^{-1}\lambda^2$.

$$L = \frac{\partial}{\partial \lambda} - \frac{(t\overline{P} - 2i\lambda R + \lambda^2 tP)}{2\lambda^2}, \quad M = \frac{\partial}{\partial t} + \frac{(\lambda\overline{P} - \lambda^3 P)}{2\lambda^2}$$

- $[L, M] = 0$ implies that

$$\begin{aligned} tP_t - i[R, P] &= 0, \\ 2iR_t + t[P, \overline{P}] &= 0. \end{aligned}$$

Bianchi V

$$
\begin{aligned}
P &= p_0(t)\,L_0 + p_1(t)\,L_1 + p_2(t)\,L_2 \\
R &= r_0(t)\,L_0 + r_1(t)\,L_1 + r_2(t)\,L_2,
\end{aligned}
$$

where $[L_0, L_1] = L_1, \quad [L_0, L_2] = L_2, \quad [L_1, L_2] = 0.$

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- $\Rightarrow \quad \dfrac{d\mathbf{r}}{dt} = t\ \mathrm{Im}(\overline{p}_0\mathbf{p}), \qquad t\dfrac{d\mathbf{p}}{dt} = i(r_0\mathbf{p} - p_0\mathbf{r}),$

$\mathbf{r} = (r_0, r_1, r_2),\ \mathbf{p} = (p_0, p_1, p_2)$

Bianchi V

$$P = p_0(t)\ L_0 + p_1(t)\ L_1 + p_2(t)\ L_2$$
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- Cf. $p_0 \neq 0, r_0 = 0 : \quad t^2\frac{d^2 r_k}{dt^2} - t\frac{dr_k}{dt} + t^2 r_k = 0, \quad k = 1, 2$

  Bessel equation

$$r_1 = c_1 t J_1(t) + c_2 t Y_1(t), \quad r_2 = c_3 t J_1(t) + c_4 t Y_1(t)$$

$$p_1 = i(c_1 J_0(t) + c_2 Y_0(t)), \quad p_2 = i(c_3 J_0(t) + c_4 Y_0(t))$$

Scalar-flat Kähler metric: $g_K = \Omega^2 g$

$$g_K = \frac{1}{4}(d\rho^2 + \rho^2 dt^2) + G_{AB}(t)dx^A dx^B, \qquad A, B = 1, 2$$

where $$G(t) = \frac{\pi^2 t^2}{16}\begin{pmatrix} Y_0^2 + Y_1^2 & -J_0 Y_0 - J_1 Y_1 \\ -J_0 Y_0 - J_1 Y_1 & J_0^2 + J_1^2 \end{pmatrix}.$$

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Theorem [Dunajski, PP (2011)]

Any Bianchi V ASD conformal structure which admits a Kähler metric s.t. the group acts holomorphically can be locally represented by a Bianchi V metric

$$g = e^1 \odot \overline{e}^1 + e^2 \odot \overline{e}^2,$$

where the one–forms e^1, e^2 are dual to

$$E_1 = \partial_t - \frac{i}{t}(r_0 L_0 + r_1 L_1 + r_2 L_2), \quad E_2 = p_0 L_0 + p_1 L_1 + p_2 L_2$$

Conclusions

- Construct ASD Bianchi V conformal classes, each of which admits a Kähler metric.
  - Obtain an explicit example of non-diagonal non-conformally flat Bianchi V metrics in terms of Bessel functions.
  - The Kähler metric is not Bianchi V invariant.