From dynamical models to cosmological observables

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Cosmological theory

The Einstein-Hilbert action for the gravitational field:

$$S_g = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,R$$

non-minimally coupled scalar field

$$S_\phi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\Big\{\varepsilon\Big(\nabla^\alpha\phi\,\nabla_\alpha\phi + \xi R\phi^2\Big) + 2U(\phi)\Big\}$$

and for the barotropic matter content

$$S_m = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_m.$$

Field Equations

The Einstein field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 \left(T^{(\phi)}_{\mu\nu} + T^{(m)}_{\mu\nu} \right) \tag{1}$$

where the energy momentum tensor for non-minimally coupled scalar field is

$$\begin{aligned} T^{(\phi)}_{\mu\nu} =& \varepsilon \nabla_\mu \phi \nabla_\nu \phi - \varepsilon \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi - U(\phi) g_{\mu\nu} \\ & + \varepsilon \xi \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \phi^2 + \varepsilon \xi \left(g_{\mu\nu} \Box \phi^2 - \nabla_\mu \nabla_\nu \phi^2 \right) \end{aligned} \tag{2}$$

and the energy momentum tensor for the barotropic matter content

$$T^{(m)}_{\mu\nu} = \left(\rho_m + p_m \right) u_\mu u_\nu + p_m g_{\mu\nu}. \tag{3}$$

FRW metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{4}$$

the energy conservation condition

$$\frac{3}{\kappa^2}H^2 = \varepsilon\frac{1}{2}\dot{\phi}^2 + U(\phi) + \varepsilon 3\xi H^2\phi^2 + \varepsilon 3\xi H(\phi^2)\dot{} + \rho_m \tag{5}$$

the acceleration equation

$$\dot{H} = -2H^2 + \frac{\kappa^2}{1 - \varepsilon\xi(1-6\xi)\kappa^2\phi^2}\Bigg(-\varepsilon\frac{1}{6}(1-6\xi)\dot{\phi}^2 + \frac{2}{3}U(\phi) - \\ - \xi\phi U'(\phi) + \frac{1}{6}(1-3w_m)\rho_m\Bigg) \tag{6}$$

the dynamical equation for the scalar field

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \varepsilon U'(\phi) = 0 \tag{7}$$

Model with a constant potential

The energy conservation condition

$$\frac{3}{\kappa^2}H^2 = \varepsilon\frac{1}{2}\dot{\phi}^2 + V_0 + \varepsilon 3\xi H^2\phi^2 + \varepsilon 6\xi H\phi\dot{\phi} + \rho_m \qquad (8)$$

where ρ_m – dust matter

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\Lambda,0} + \Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \varepsilon(1-6\xi)x^2 + \varepsilon 6\xi(x+z)^2 \qquad (9)$$

where

$$\Omega_{\Lambda,0} \equiv \frac{\kappa^2 V_0}{3H_0^2}, \qquad \Omega_{m,0} \equiv \frac{\kappa^2\rho_{m,0}}{3H_0^2}$$

and the new phase space variables

$$x \equiv \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H_0}, \qquad z \equiv \frac{\kappa}{\sqrt{6}}\frac{H}{H_0}\phi$$

The acceleration equation

$$\frac{\dot{H}}{H^2} = -2 + \frac{-\varepsilon(1-6\xi)x^2 + 2\Omega_{\Lambda,0} + \frac{1}{2}\Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3}}{\left(\frac{H}{H_0}\right)^2 - \varepsilon 6\xi(1-6\xi)z^2}.$$

Dynamical system

The dynamical system on variables x, z and $h = \frac{H}{H_0}$ is in the following form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\ln a} &= -3x - 6\xi z\left(\frac{\dot{H}}{H^2} + 2\right), \\ \frac{\mathrm{d}z}{\mathrm{d}\ln a} &= x + z\frac{\dot{H}}{H^2}, \\ \frac{\mathrm{d}h}{\mathrm{d}\ln a} &= h\frac{\dot{H}}{H^2}, \end{aligned} \tag{10}$$

where

$$\frac{\dot{H}}{H^2} = -2 + \frac{3}{2}\frac{\Omega_{\Lambda,0} + \frac{1}{3}h^2 - \varepsilon(1-6\xi)x^2 - \varepsilon 2\xi(x+z)^2}{h^2 - \varepsilon 6\xi(1-6\xi)z^2}. \tag{11}$$

From the last equation, one can notice, that the system has two invariant submanifolds $h = H/H_0 = 0$ and $\dot{H}/H^2 = 0$. This second property will be used later in the paper in order to obtain an explicit solution of a restricted dynamical system.

Two special cases

For $\xi = 0$ (minimal coupling) we have

$$\left(\frac{H(a)}{H(a_0)}\right)^2\Bigg|_{\xi=0} = \Omega_{\Lambda,0} + \Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \varepsilon\, x(a)^2$$

and the system (10) reduces to single equation

$$\frac{\mathrm{d}x}{\mathrm{d}\ln a} = -3x\,,$$

which can be easily integrated

$$x(a) = x(a_0)\left(\frac{a}{a_0}\right)^{-3}$$

$$\left(\frac{H(a)}{H(a_0)}\right)^2\Bigg|_{\xi=0} = \Omega_{\Lambda,0} + \Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \varepsilon\, x_0^2\left(\frac{a}{a_0}\right)^{-6} \qquad (12)$$

Two special cases

For $\xi = \frac{1}{6}$ (conformal coupling) we have

$$\left(\frac{H(a)}{H(a_0)}\right)^2\Bigg|_{\xi=\frac{1}{6}} = \Omega_{\Lambda,0} + \Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \varepsilon\,(x(a)+z(a))^2$$

and the system (10) reduces to

$$\frac{\mathrm{d}(x+z)}{\mathrm{d}\ln a} = -2(x+z)$$

which can be integrated

$$x(a)+z(a) = (x(a_0)+z(a_0))\left(\frac{a}{a_0}\right)^{-2}$$

$$\left(\frac{H(a)}{H(a_0)}\right)^2\Bigg|_{\xi=\frac{1}{6}} = \Omega_{\Lambda,0} + \Omega_{m,0}\left(\frac{a}{a_0}\right)^{-3} + \varepsilon\,(x_0+z_0)^2\left(\frac{a}{a_0}\right)^{-4} \quad (13)$$

Invariant manifold

On the invariant manifold $\frac{\dot{H}}{H^2} = 0$ the system (10) reduces to

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\ln a} &= -3x - 12\xi z\,, \\ \frac{\mathrm{d}z}{\mathrm{d}\ln a} &= x\,. \end{aligned} \tag{14}$$

This linear system of equations can be directly integrated.
We make the following change of variables

$$\begin{aligned} x &= \lambda_1 p + \lambda_2 q\,, \\ z &= p + q\,, \end{aligned} \tag{15}$$

where $\lambda_1 = -\frac{3}{2} - \frac{1}{2}\sqrt{3(3-16\xi)}$, $\lambda_2 = -\frac{3}{2} + \frac{1}{2}\sqrt{3(3-16\xi)}$

Invariant manifold

We obtain

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}\ln a} &= \lambda_1 p\,, \\ \frac{\mathrm{d}q}{\mathrm{d}\ln a} &= \lambda_2 q\,. \end{aligned} \tag{16}$$

and the solutions are

$$\begin{aligned} p(a) &= p(a_0)\left(\frac{a}{a_0}\right)^{\lambda_1}\,, \\ q(a) &= q(a_0)\left(\frac{a}{a_0}\right)^{\lambda_2}\,. \end{aligned} \tag{17}$$

The Hubble function

The acceleration equation

$$\frac{\mathrm{d}h}{\mathrm{d}\ln a} = h\left(-2 + \frac{3}{2}\frac{\Omega_{\Lambda,0} + \frac{1}{3}h^2 - \varepsilon(1-6\xi)x(a)^2 - \varepsilon 2\xi(x(a)+z(a))^2}{h^2 - \varepsilon 6\xi(1-6\xi)z(a)^2}\right)$$

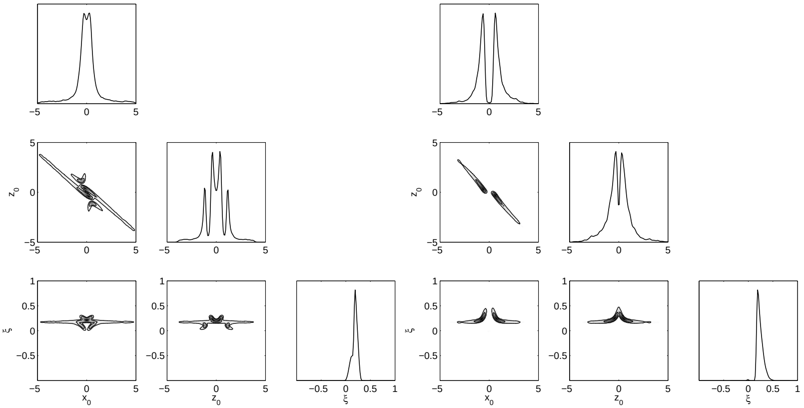
where

$$x(a) = \lambda_1 p(a) + \lambda_2 q(a)\,,$$
$$z(a) = p(a) + q(a)\,,$$

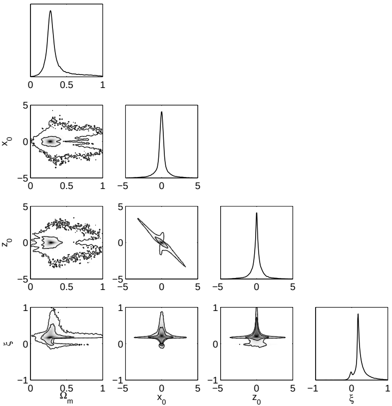
and

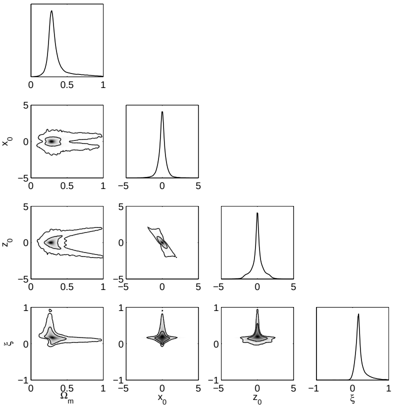
$$\Omega_{\Lambda,0} = 1 - \Omega_{m,0} - \varepsilon(1-6\xi)x_0^2 - \varepsilon 6\xi(x_0+z_0)^2\,.$$

Canonical and phantom scalar field + $\Omega_{m,0} = \Omega_{bm,0}$ fixed



Canonical and phantom scalar field + $\Omega_{m,0}$ free





Analysis at infinity

In order to obtain complete information about the structure of the phase space of the dynamical system it is necessary to investigate the behavior of the trajectories of this system at infinity. We need to introduce three charts $U_{(n)}$ of projective coordinates :

$$
\begin{aligned}
U_{(1)}: &\quad u_{(1)} = \frac{1}{x}\,, \; v_{(1)} = \frac{z}{x}\,, \; w_{(1)} = \frac{h}{x}\,, \\
U_{(2)}: &\quad u_{(2)} = \frac{x}{z}\,, \; v_{(2)} = \frac{1}{z}\,, \; w_{(2)} = \frac{h}{z}\,, \\
U_{(3)}: &\quad u_{(3)} = \frac{x}{h}\,, \; v_{(3)} = \frac{z}{h}\,, \; w_{(3)} = \frac{1}{h}\,.
\end{aligned}
$$

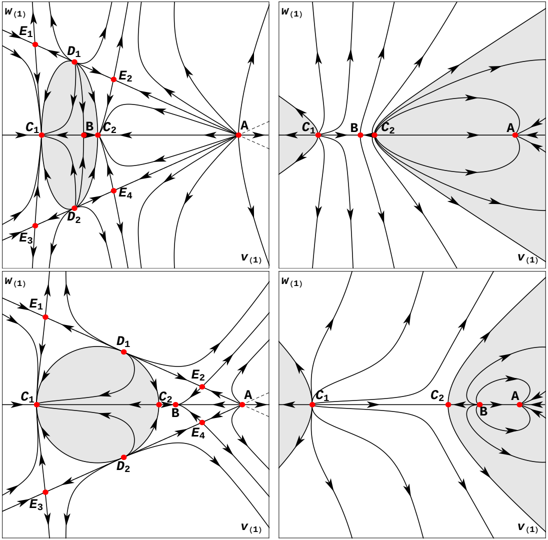


Figure : $\varepsilon = -1$, $\frac{1}{6} < \xi < \frac{3}{16}$ (top left), $\varepsilon = +1$, $\frac{1}{6} < \xi < \frac{3}{16}$ (top right), $\varepsilon = -1$, $\frac{3}{16} < \xi < \frac{1}{4}$ (bottom left), $\varepsilon = +1$, $\frac{3}{16} < \xi < \frac{1}{4}$ (bottom right)

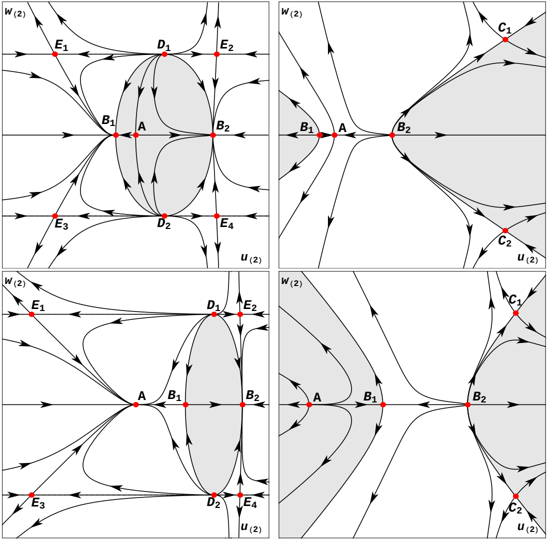
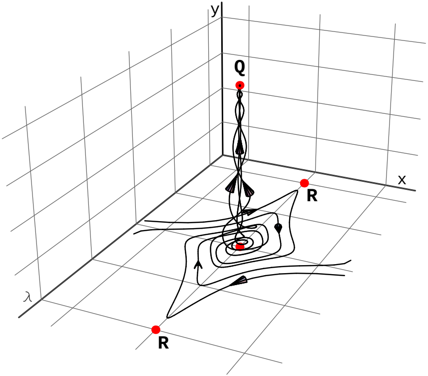


Figure : $\varepsilon = -1$, $\frac{1}{6} < \xi < \frac{3}{16}$ (top left), $\varepsilon = +1$, $\frac{1}{6} < \xi < \frac{3}{16}$ (top right), $\varepsilon = -1$, $\frac{3}{16} < \xi < \frac{1}{4}$ (bottom left), $\varepsilon = +1$, $\frac{3}{16} < \xi < \frac{1}{4}$ (bottom right)

Twister Quintessence Scenario



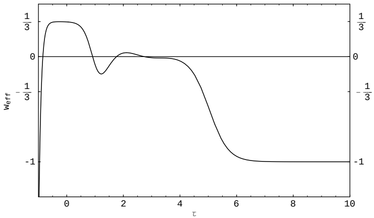


Figure : $x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H}$, $y \equiv \frac{\kappa\sqrt{U(\phi)}}{\sqrt{3}H}$, $\lambda \equiv -\frac{\sqrt{6}}{\kappa}\frac{U'(\phi)}{U(\phi)}$

Conclusions and outlook

- Observational constrains on dynamics of the model on the invariant manifold of the de Sitter state suggest $\xi \approx \frac{3}{16}$ independently of the scalar field.
- Dynamical system methods suggest the same value for a transient inflationary epoch.
- Do the inflation works in the model ?
- What about constrains from CMB ?