Hilbert space & polymer matter in quantum cosmology

1ST CONFERENCE OF POLISH SOCIETY IN RELATIVITY

SPAŁA, POLAND, 29.06-4.07.2014

Tomasz Pawłowski

Departmento de Ciencias Físicas
Universidad Andrés Bello, Santiago, Chile

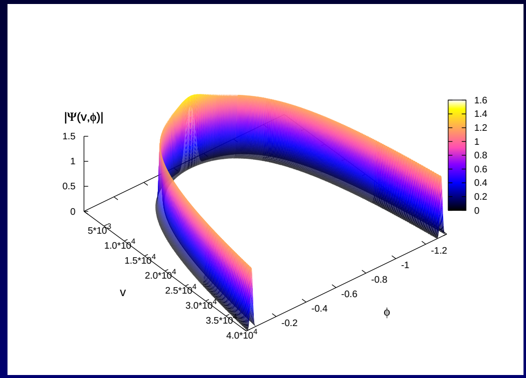
work by:

A. Kreienbuehl, TP (2013)

F. Barbero, TP, E. Villaseñor (2014)

Motivation

- Polymer quantization of geometry degrees of freedom in LQC $\longrightarrow$ qualitative changes of dynamics at Planck energies (big bounce).



- Most works: only matter quantized polymerically, matter treated in standard way.
- Consistency requires same treatment of geometry and matter:
  - In many cases 'matter' represents geometry DOF (inhomogeneities in hybird quantization).
  - Often the framework mixes physical geometry and matter DOF (nonminimal coupling in Einstein frame).
- Expectation: New quanitative changes to dynamics.

UNIVERSIDAD ANDRES BELLO

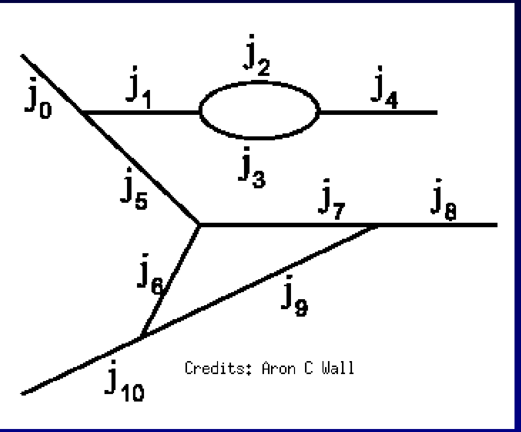
The work

- Testing the expectations (the dynamics) and the framework constructions on a simple model.
- The model:
  - Flat FLRW (isotropic) universe with massless scalar field.
  - Scalar field quantized polymerically.
  - To distinguish polymer matter effects the geometry quantized via geometrodynamics.
- Unexpected issue: viable construction of the correct separable Hilbert space for the model.
  - Issue critical to have dynamics with correct GR limit!
  - Solution: Fiber integral Hilbert space
    - Construction presented on example of polymer harmonic oscillator.

UNIVERSIDAD ANDRES BELLO

Hilbert spaces in LQG & LQC

- In LQC basis of either kinematical or diff-invariant Hilbert space: cyllindrical functions supported on the graphs - spin networks
  - Disjoint graphs - mutually orthogonal.
  - On each single graph: space is separable, but
  - Continuum of graphs - nonseparability
    - Lots of useful mathematical structure lost/ requires extension.



- In simplified QM models (i.e. LQC) the same situation.
  - Example: Isotropic LQC – $\mathcal{H}_{\rm gr} = L^2(\mathbb{R}_{\rm Bohr}, {\rm d}\mu_{\rm Bohr}) \leftrightarrow \Sigma^2(\mathbb{R})$.
    - Discrete inner product - space nonseparable.
    - In considered scenario (polymer scalar): following standard constructions (shadow states) – nonphysical dynamics.
- Question: is there a natural construction of alternative – separable spaces?
- Goal: Providing such construction in simple QM models.

(Loopy) harmonic oscillator

F. Barbero, J. Prieto, E. Villaseñor, 2013

- Classical theory:
  - Canonical variables: q, p.
  - Hamiltonian: $H(q,p) = \frac{\hbar^2}{2m\ell^2}p^2 + \frac{m\ell^2\omega^2}{2}q^2$.
  - Weyl algebra: $e^{i\lambda q}$, $e^{i\sigma p}$.
- Polymer quantization: Analog of LQC quantization program.
  - Roles of q and p interchangeable.
  - Equivalent of holonomy-flux algebra: either $e^{i\lambda q}$ or $e^{i\sigma p}$
    - Choice: $e^{i\sigma p}$ – almost periodic functions.
  - Gel'fand spectrum (support of $\mathcal{H}$): Bohr compactification of $\mathbb{R}$.
  - Hilbert space: $\mathcal{H} = L^2(\bar{\mathbb{R}}, \mathrm{d}\mu)$.
  - Operators: $\hat{q}$, $\widehat{e^{i\sigma p}}$, $\hat{p}$ doesn't exist!

(Loopy) harmonic oscillator

- Analog of LQC regularization: Regularized quantum Hamiltonian

$$\hat{H} = \frac{\hbar^2}{8mq_o^2\ell^2}(2\mathbb{I} - \hat{N}^2 - \hat{N}^{-2}) + \frac{m\ell^2\omega^2}{2}\hat{q} \qquad N = e^{-q_o p}$$

- Action of the Hamiltonian:
  - p -representation: Particle in periodic potential

$$[\hat{H}\Psi](p) = -\frac{m\ell^2\omega^2}{2}\Psi''(p) + \frac{\hbar^2}{2mq_o^2\ell^2}\sin^2(q_o p)\Psi(p), \quad \Psi \in \mathcal{H}$$

  - q -representation: $\Psi \in \Sigma^2(\mathbb{R})$ a difference operator.

$$[\hat{H}\Psi](q) = -\frac{\hbar^2}{8mq_o^2\ell^2}[\Psi(q + 2q_o) - 2\Psi(q) + \Psi(q - 2q_o)] + \frac{m\ell^2\omega^2}{2}q\Psi(q)$$

- Treatment: exactly like system with periodic potential (std. quantization).
- Spectrum: Band structure (as for periodic pot.), but
  - Eingenfunctions explicitly normalizable wrt Bohr measure – Purely point spectrum!! - space nonseparable

UNIVERSIDAD ANDRES BELLO

The "easy" solution

Corichi, Vukasinac, Zapata, 2007

Direct repetition of treatment in isotropic LQC.

- Observation: In q-rep. $\hat{H}$ is a difference operator.
- Sets preserved by action: $\mathcal{L}_\epsilon := 2q_o(\epsilon + \mathbb{Z})$ $\epsilon \in [0, 1)$.
- Division of $\mathcal{H}$ onto subspaces $\mathcal{H}_\epsilon = \mathcal{H}|_{\mathcal{L}_\epsilon}$.
- $\mathcal{H}_\epsilon$ preserved by observables $\rightarrow$ superselection sectors.
- Idea: Restrict to a single sector.
  - $\mathcal{H}_\epsilon$ are separable.
  - Spectrum Sp_ϵ of $\hat{H}$ - isolated point spectrum.
  - Union of spectra reproduces the band structure $\bigcup_\epsilon \mathrm{Sp}_\epsilon = \mathrm{Sp}(\hat{H})$
- Problem: Sectors get mixed as soon as q_o becomes time dependent.
  - Inhomogeneity modes in hybrid approach to inhomogeneous LQC.
  - Polymer massless scalar field in FRW.

UNIVERSIDAD ANDRES BELLO

The cosmological system

A. Kreienbuehl, TP, 2013

Flat FRW universe (geometrodynamics) with polymeric massless scalar field.

- Classical theory:
  - Metric: $\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2]$.
  - Variables: $\{a, \pi_a\} = 1$, $\{\phi, p_\phi\} = 1$ a-oriented.
  - Hamiltonian constraint:

$$H[N] = N\frac{V_o|a|^3}{24\pi G}\left(12\pi G\left(\frac{p_\phi}{V_o a^3}\right)^2 - \left(\frac{4\pi G\pi_a}{V_o a^2}\right)\right)$$

  - Classical deparametrization: Scalar evolving wrt scale factor.

    $t = v \propto a^3$

$$-p_t = H_s := \left|\frac{p_\phi}{t}\right|$$

    - Scalar field of time-dependent kinetic term evolving in "oriented volume" time.

Cosm. system: quantization

Deparametrization on the classical level implicitly implies Wheeler-DeWitt quantization of the geometry degrees of freedom.

- Polymer quantization of the scalar field:
  direct analog of the one for oscillator.
- Quantum counterparts of Wayl algebra elements always exist

$$\hat{N}(\lambda) := e^{\lambda p_\phi} \qquad \hat{S}(\sigma) := e^{\sigma\phi}$$

  Only one of operators $\hat{\phi}, \hat{p}_\phi$ exists!
- Hilbert space: $L^2(\bar{\mathbb{R}}, \mathrm{d}\mu)$ - Bohr compact. Nonseparable
- Two inequivalent (dual) choices: (momentum and configuration rep.)
  - The momentum rep: basic operators are momentum and boost:
    $\hat{p}_\phi$, $\hat{S}(\sigma) := e^{\sigma\phi}$
    Lewandowski, Domagała, Dzieńdzikowski, 2011
    - In both WDW and LQC (for massless field) results same as for Schrödinger quantization.

Cosm. system: quantization (2)

Our choice, the second prescription: the configuration representation:

Hossain, Husain, Seahra, 2010

- Basic operators: field and shift:

$$\hat{\phi}, \qquad \hat{N}(\lambda) := e^{\lambda p_\phi}$$

- Evolution: Schrödinger equation (Hamiltonian regularized)

$$i\partial_t \Psi = \hat{H}\Psi = \frac{i}{2t\lambda}[N(\lambda) - N^{-1}(\lambda)]\Psi$$

- Consistency: invariance wrt. change of the (infrared regulating) fiducial cell implies time-dependent polymerization scale:

$$\lambda = \lambda_o/|t|$$

- At given instant of time one can introduce the superselection sectors supported on “lattices” preserved by action of $\hat{H}(t)$, but lattice spacing is time-dependent.
  - Evolution mixes the “instantaneous” superselection sectors.

Cosm. system: evolution

- Evolution:
  - $\hat{p}_\phi$ does not exist but p_ϕ can be still used as a coordinate
  - in momentum representation Schrödinger equation becomes p_ϕ-parametrized ODE.
  - The solution:

$$\Psi(t, p_\phi) = e^{i(F(t,p_\phi) - F(t_o,p_\phi))}\Psi(t_o, p_\phi)$$

$$F(v, p_\phi) := v\left[\sin(p_\phi/v) - (p_\phi/v)\mathrm{Ci}(p_\phi/v)\right]$$

    - Not an almost periodic function!
  - Shadow state formalism: Ashtekar, Fairhurst, Willis, 2003
    - Take dual space and adjoint of $\hat{H}$.
    - Project onto basis of almost periodic f. $(\Psi \hat{H} \psi\rangle$
  - The trajectory: evaluating expectation values

$$\phi(t) = \langle\hat{\phi}\rangle(t) = \phi(t_o)$$

    - Naive evolution frozen: Inconsistent with GR!

UNIVERSIDAD ANDRES BELLO

Hilbert sp.: the proposal

Back to loop quantum harmonic oscillator.

- Observation for the case of Schrödinger quantization of a particle in periodic potentials:
  - There exists a fiber bundle decomposition of $\mathcal{H}$ onto Hilbert spaces $\mathcal{H}_\epsilon$ of functions supported on $\mathcal{L}_\epsilon$.

$$\text{“}\ \mathcal{H} = \int \mathrm{d}\sigma(\epsilon)\mathcal{H}_\epsilon\ \text{”}$$

- In loop quantization: We no longer have the decomposition but we still have the “single fibers” – spaces corresponding to each single superselection sector.
- Idea: Build back a Hilbert space out of superselection sector subspaces.
  - Construct a Hilbert space for which the subspaces $\mathcal{H}_\epsilon$ will be elements of a fiber bundle decomposition.

UNIVERSIDAD ANDRES BELLO

Hilbert sp.: the construction

F. Barbero, TP, E. Villaseñor, 2014

- Motivation: Structure of self-adjoint extensions for flat FRW with $+$ve cosmological constant in LQC. W. Kamiński, J. Lewandowski, TP, 2009
- Construction:
  - $\mathcal{H}_\epsilon$ separable
  - Lebesgue measure on the set of supersel. sect. labels ϵ.
  - Can build an "integral over sectors".

$$\tilde{\mathcal{H}} := \int_{[0,1)} \mathrm{d}\epsilon \mathcal{H}_\epsilon \quad \langle \Psi | \Psi' \rangle_{\tilde{\mathcal{H}}} = \int_{[0,1)} \langle \Psi_\epsilon | \Psi'_\epsilon \rangle_{\mathcal{H}_\epsilon} \quad \Psi_\epsilon := \Psi|_{\mathcal{L}_\epsilon}$$

- Properties:
  - $\tilde{\mathcal{H}}$ is separable.
  - The spectrum of $\hat{H}$ has the same band structure, but now is purely continuous.
- Result: loop harmonic oscilator looks like a particle in periodic potential.

UNIVERSIDAD ANDRES BELLO

Application to our system

We apply the integral $\mathcal{H}$ construction to retrieve the correct dynamics

- Instantaneous superselection sectors:

$$\phi \in \mathcal{L}_\epsilon = \lambda_o(\epsilon + \mathbb{Z})/|v|$$

  - The spaces: $\mathcal{H}_\epsilon(v)$: restr. to functions supported on $\mathcal{L}_\epsilon$ (at given v).

- The integral space: $\tilde{\mathcal{H}} := \int_0^1 \mathcal{H}_\epsilon(v) M(\epsilon, v)$

- Inner product: integral over IP of "fibers"

$$\langle \Psi | \Psi' \rangle = \int \bar{\Psi}(v,\phi)\Psi'(v,\phi)M(v,\epsilon(\phi))/\lambda(v)\mathrm{d}\phi$$

- Time dependence of M fixed by requirement of unitarity of evolution.

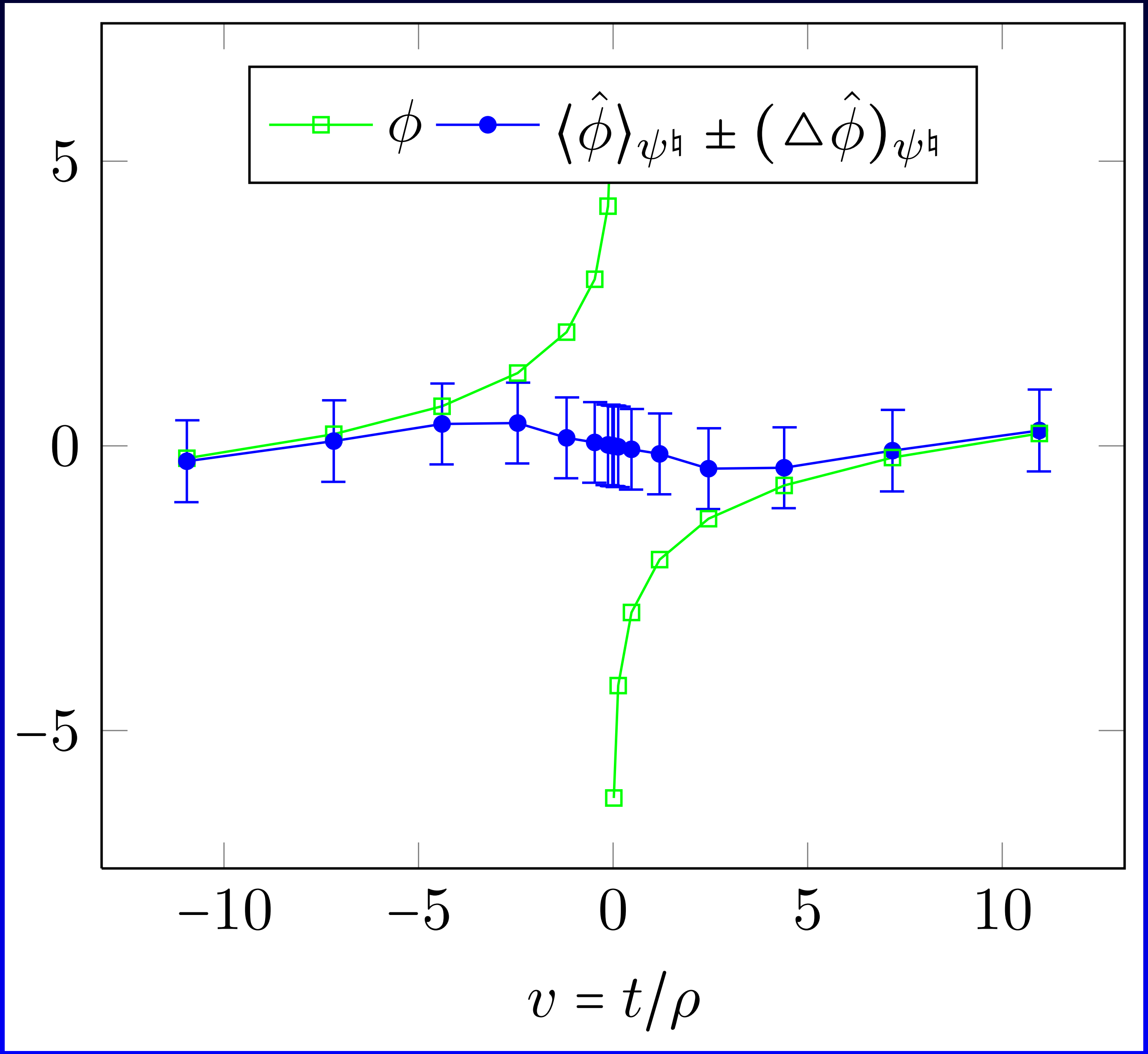
- Final result: Space unitarily equivalent to

$$\tilde{H} \equiv L^2(\mathbb{R}, \mathrm{d}\phi)$$

- Dynamics: evaluating again $\phi(t) = \langle \hat{\phi} \rangle (t)$
  - trajectory regular at $v = 0$ and approaching GR at large v

UNIVERSIDAD ANDRES BELLO

Cosm. system: the dynamics



Conclusions

- The polymer quantization (used in LQC) can be easily extended to matter fields. (already known).
- For scalar field: there are several ambiguities in implementing the quantization.
- One of choices (regularized momentum) applied to flat FRLW universe with massless scalar field (Schrödinger quantum geometry) leads to a significant modifications of dynamics near the classical singularity.
  - The trajectory becomes regular (finite $\langle \phi(t) \rangle$ at $v = 0$).
  - Deterministic (unique) extension of the dynamics past $v = 0$.
  - Retrieved the correct GR limit at low energy scales.
- In the process we have to abandon standard construction of a Hilbert space
  - Convenient construction of a separable space as a "fiber integral", applicable to LQC and beyond.

UNIVERSIDAD ANDRES BELLO

Thank you for your attention!

Research and presentation supported in part by National Center for Science (NCN), Poland under grant no. 2012/05/E/ST2/03308 and the Chilean FONDECYT regular grant no.1140335.

UNIVERSIDAD ANDRES BELLO