Post-Newtonian Hamiltonian dynamics of compact binary systems

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- P.J. and G.Schäfer, Phys. Rev. D **86**, 061503(R) (2012)
- P.J. and G.Schäfer, Phys. Rev. D **87**, 081503(R) (2013)
- T.Damour, P.J., and G.Schäfer, Phys. Rev. D **89**, 064058 (2014)

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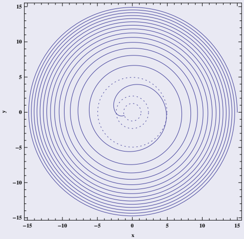
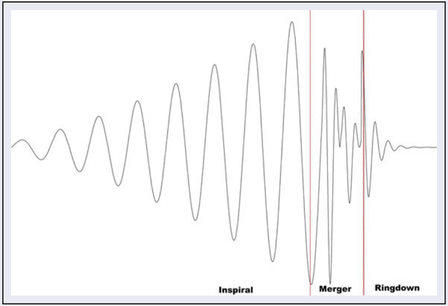
  is interesting and difficult.
- Relativistic binary systems exist in nature,
  they comprise compact objects: neutron stars or black holes.
- These systems emit gravitational waves,
  which experimenters try to detect
  within the LIGO/VIRGO/GEO600 projects.

Black-hole binary: Inspiral—merger—ringdown

As the result of emission of gravitational waves,
the size of the binary member orbits decrease
and the binary components move faster,
consequently leading to emission of gravitational waves
with increasing amplitude and frequency
—producing a gravitational-wave chirp signal.





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- One has to construct a discrete bank of templates (waveforms): templates are parametrized by possible values of the GW signal's parameters (e.g. masses and spins of binary components).
- For construction of templates one has to know time evolution of the source of gravitational waves (i.e. compact binary system in the rest of this talk).

Different approaches for solving relativistic two-body problem

- Numerical relativity (breakthrough in 2005).
- Approximate 'analytical' methods:
  - post-Newtonian expansion,
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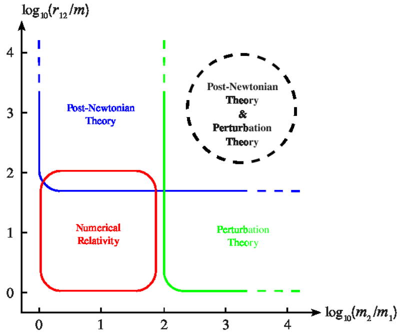
(Blanchet et al., Phys. Rev. D **81**, 064004 (2010))

- PN expansion:
  0th order—Newtonian gravity;
  nPN order—corrections of order

$$\left(\frac{v}{c}\right)^{2n} \sim \left(\frac{Gm}{rc^2}\right)^n$$

  to the Newtonian gravity.
- Perturbation approach: $\frac{m_2}{m_1} \gg 1$.



Computing power constraints and hybrid templates

- For GW signals coming from coalescence of (especially) spinning binary black holes (BBHs) with arbitrary mass ratios, due to limitations in available computing power, it will not be possible in the nearest future to construct bank of templates based purely on numerical results.

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- Gravitational-wave induced evolution of BBH computed numerically and by means of the PN approximations agree very well in the region, where the coalescing objects are sufficiently far away.

Computing power constraints and hybrid templates

- For GW signals coming from coalescence of (especially) spinning binary black holes (BBHs) with arbitrary mass ratios, due to limitations in available computing power, it will not be possible in the nearest future to construct bank of templates based purely on numerical results.
- Gravitational-wave induced evolution of BBH computed numerically and by means of the PN approximations agree very well in the region, where the coalescing objects are sufficiently far away.
- Detection of GW signals (and extraction their parameters) from coalescing BBH by advanced LIGO/VIRGO detectors —the most promising templates are hybrid templates: PN (early inspiral) + numerical (late inspiral, merger, ringdown).

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EOM	$\boxed{\text{N}}$	$\boxed{\text{1PN}}$	$\boxed{\text{2PN}}$	$\boxed{\text{2.5PN}}$	$\boxed{\text{3PN}}$	$\boxed{\text{3.5PN}}$	$\boxed{\text{4PN}}$	4.5PN	5PN	5.5PN	6PN	6.5PN
Luminosity	—	—	—	$\boxed{\text{N}}$	—	$\boxed{\text{1PN}}$	$\boxed{\text{1.5PN}}$	$\boxed{\text{2PN}}$	$\boxed{\text{2.5PN}}$	$\boxed{\text{3PN}}$	$\boxed{\text{3.5PN}}$	4PN

The laser-interferometric detector response function

The response function of the laser-interferometric detector to gravitational waves from coalescing compact binary (made of nonspinning bodies) in circular orbits:

$$h(t) = \frac{C}{D}\big(\dot{\phi}(t)\big)^{2/3}\sin\big(2\phi(t)+\alpha\big),$$

where $\phi(t)$ is the orbital phase of the binary
(so $\dot{\phi}(t) \equiv \mathrm{d}\phi(t)/\mathrm{d}t$ is the angular frequency),
D is the distance of the binary to the Earth,
C and α are some constants.

The orbital phase $\phi(t)$ evolution is computed from the balance equation

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{L},}$$

which has the following PN expansion:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(E_{\mathrm{N}} + \frac{1}{c^2}E_{1\mathrm{PN}} + \frac{1}{c^4}E_{2\mathrm{PN}} + \frac{1}{c^6}E_{3\mathrm{PN}} + \frac{1}{c^8}E_{4\mathrm{PN}} + \mathcal{O}\big((v/c)^9\big)\Big)\\ = -\Big(\mathcal{L}_{\mathrm{N}} + \frac{1}{c^2}\mathcal{L}_{1\mathrm{PN}} + \frac{1}{c^3}\mathcal{L}_{1.5\mathrm{PN}} + \frac{1}{c^4}\mathcal{L}_{2\mathrm{PN}} + \frac{1}{c^5}\mathcal{L}_{2.5\mathrm{PN}}\\ + \frac{1}{c^6}\mathcal{L}_{3\mathrm{PN}} + \frac{1}{c^7}\mathcal{L}_{3.5\mathrm{PN}} + \frac{1}{c^8}\mathcal{L}_{4\mathrm{PN}} + \mathcal{O}\big((v/c)^9\big)\Big).\end{aligned}$$

Masses of the bodies: $m_1,\ m_2,$

$$M \equiv m_1 + m_2, \quad \mu \equiv \frac{m_1 m_2}{M},$$

$$\nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}, \quad 0 \le \nu \le \frac{1}{4}.$$

Dimensionless post-Newtonian parameter for circular orbits:

$$x \equiv \frac{1}{c^2}(GM\dot{\phi})^{2/3}.$$

4PN-accurate bounding energy in the center-of-mass frame for circular orbits

$$E(x;\nu) = -\frac{\mu c^2 x}{2}\bigg(1 + e_{1\mathrm{PN}}(\nu)\,x + e_{2\mathrm{PN}}(\nu)\,x^2 + e_{3\mathrm{PN}}(\nu)\,x^3 + \Big(e_{4\mathrm{PN}}(\nu) + \frac{448}{15}\nu\ln x\Big)\,x^4 + \mathcal{O}\big((v/c)^{10}\big)\bigg),$$

$$e_{1\mathrm{PN}}(\nu) = -\frac{3}{4} - \frac{1}{12}\nu, \qquad e_{2\mathrm{PN}}(\nu) = -\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2,$$

$$e_{3\mathrm{PN}}(\nu) = -\frac{675}{64} + \Big(\frac{34445}{576} - \frac{205}{96}\pi^2\Big)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3,$$

$$e_{4\mathrm{PN}}(\nu) = -\frac{3969}{128} + c_1\,\nu \boxed{+ \Big(-\frac{498449}{3456} + \frac{3157}{576}\pi^2\Big)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4}$$

(Jaranowski & Schäfer 2012–2013, Foffa & Sturani 2013),

$$c_1 = -\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}(2\ln 2 + \gamma))$$ (γ is the Euler's constant)

(Le Tiec, Blanchet, & Whiting 2012, Bini & Damour 2013).

Gravitational-wave luminosity for circular orbits up to the 3.5PN order

$$\mathcal{L}(x;\nu) = \frac{32c^5}{5G}\nu^2 x^5 \Big(1 + \ell_{1\mathrm{PN}}(\nu)\,x + 4\pi\,x^{3/2} + \ell_{2\mathrm{PN}}(\nu)\,x^2 + \ell_{2.5\mathrm{PN}}(\nu)\,x^{5/2} + \Big(\ell_{3\mathrm{PN}}(\nu) - \frac{856}{105}\ln(16x)\Big)\,x^3 + \ell_{3.5\mathrm{PN}}(\nu)\,x^{7/2} + \mathcal{O}\big((v/c)^8\big)\Big),$$

$$\ell_{1\mathrm{PN}}(\nu) = -\frac{1247}{336} - \frac{35}{12}\nu, \qquad \ell_{2\mathrm{PN}}(\nu) = -\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2,$$

$$\ell_{2.5\mathrm{PN}}(\nu) = \Big(-\frac{8191}{672} - \frac{535}{24}\nu\Big)\pi,$$

$$\ell_{3\mathrm{PN}}(\nu) = \frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma + \Big(-\frac{134543}{7776} + \frac{41}{48}\pi^2\Big)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3,$$

$$\ell_{3.5\mathrm{PN}}(\nu) = \Big(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\Big)\pi.$$

Why Dirac deltas?

Dirac-delta distributions δ lead to self-field divergencies—regularization is needed.

♣ Usage of δs simplifies computations.

♣ **Modelling source terms for black-hole spacetimes**: the Brill-Lindquist solution of time-symmetric two-black-hole initial value problem (Jaranowski & Schäfer, 1999).

♣ **Effacement principle** (Damour, 1983): dimensions and internal structure of the *compact* and *nonrotating* bodies enter their equations of motion only at the 5PN order.

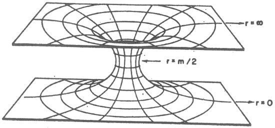

FIG. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2=(1+m/2r)^4(dr^2+r^2d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold ($r\to 0$, $r\to\infty$).

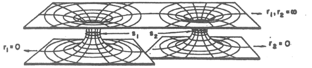

FIG. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

$$ds^2=(1+m/2r_1+m/2r_2)^4 ds_F^2.$$

Point masses together with dimensional regularization give unique equations of motion (EOM) and gravitational-wave luminosities up to the 3.5PN order.

E.g., there exist three independent derivations, using δs, of the 3PN conservative 2-point-mass EOM:

(i) ADM-Hamiltonian-based derivation (Damour, Jaranowski, Schäfer, 1998–2001);

(ii) harmonic-coordinate-based derivation (Blanchet, Damour, Esposito-Farèse, Faye, 2000–2004);

(iii) effective-field-theory approach (Foffa, Sturani, 2011).

- 2PN-accurate EOM were derived for extended body models by Grishchuk & Kopeikin (1985–1986), the result is compatible with 2PN EOM derived for the δ-sources.
- There exists fourth independent derivation of the 3PN EOM in harmonic coordinates using a surface-integral approach (Itoh, Futamase, 2003–2004), the result is compatible with 3PN EOM derived for the δ-sources.

Units/notation

Units: $c = 16\pi G_{d+1} = 1$ (quite often).

Asymptotically flat $(d+1)$-dimensional spacetime,
asymptotically Minkowskian reference frame
with coordinates $x^0 \equiv t$, $\mathbf{x} \equiv (x^1, \ldots, x^d)$.

Particle labels: $a, b \in \{1, 2\}$.

Mass parameters of the particles: m_a,

position vectors of the particles: $\mathbf{x}_a \equiv (x_a^1, \ldots, x_a^d)$,

momentum vectors of the particles: $\mathbf{p}_a \equiv (p_{a1}, \ldots, p_{ad})$.

Notation (cont.)

For any d-vectors $\mathbf{v} = (v^1, \ldots, v^d)$ and $\mathbf{w} = (w^1, \ldots, w^d)$:

$$\mathbf{v} \cdot \mathbf{w} := \delta_{ij} v^i w^j, \quad |\mathbf{v}| := \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

$$\mathbf{r}_a := \mathbf{x} - \mathbf{x}_a, \quad r_a := |\mathbf{r}_a|, \quad \mathbf{n}_a := \mathbf{r}_a / r_a;$$

$$\text{for } a \neq b\text{:} \quad \mathbf{r}_{ab} := \mathbf{x}_a - \mathbf{x}_b, \quad r_{ab} := |\mathbf{r}_{ab}|, \quad \mathbf{n}_{ab} := \mathbf{r}_{ab} / r_{ab}.$$

Canonical variables

Canonical matter variables:

$$\left.\begin{aligned}\mathbf{x}_a &= (x_a^1, \dots, x_a^d),\\ \mathbf{p}_a &= (p_{a1}, \dots, p_{ad}),\end{aligned}\right\} \quad a = 1, 2.$$

Canonical field variables:

$$\gamma_{ij} := g_{ij},$$

$$\pi^{ij} := \sqrt{\gamma}(K^{ij} - \gamma^{ij}\gamma^{kl}K_{kl}),$$

K_{ij} is the extrinsic curvature of the hypersurface $x^0 = const$.

Constraint equations for 2-point-mass systems

$$\sqrt{\gamma}\,R - \frac{1}{\sqrt{\gamma}}\left(\gamma_{ik}\,\gamma_{j\ell}\,\pi^{ij}\,\pi^{k\ell} - \frac{(\gamma_{ij}\,\pi^{ij})^2}{d-1}\right) = \sum_{a=1}^{2}\sqrt{\boxed{\gamma_a^{ij}}\,p_{ai}p_{aj} + m_a^2}\,\delta(\mathbf{x}-\mathbf{x}_a),$$

$$-2\left(\pi^{ij}{}_{,j} + \Gamma^i_{jk}\pi^{jk}\right) = \sum_{a=1}^{2}\boxed{\gamma_a^{ij}}\,p_{aj}\,\delta(\mathbf{x}-\mathbf{x}_a),$$

$\gamma_a^{ij} \equiv \gamma^{ij}(\mathbf{x}_a)$ is perturbatively unambigously defined and finite (we expect that at least up to the 4PN order).

Point-particle energy momentum tensor

Source terms for the constraint equations are derived from the energy momentum tensor

$$T^{\alpha\beta}(x^\mu) := \sum_{a=1}^{2} m_a \int_{-\infty}^{+\infty} \frac{u_a^\alpha u_a^\beta}{\sqrt{-\det(g_{\mu\nu})}} \delta^{d+1}\big(x^\mu - z_a^\mu(\tau_a)\big)\,\mathrm{d}\tau_a,$$

τ_a is the proper time along the world line $x^\mu = z_a^\mu(\tau_a)$ of the ath particle, and

$$u_a^\alpha := \frac{\mathrm{d}z_a^\alpha}{\mathrm{d}\tau_a}.$$

The ADM transverse-traceless (TT) gauge

$$\gamma_{ij} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{4/(d-2)} \delta_{ij} + h_{ij}^{\mathrm{TT}},$$

$$\pi^{ii} = 0.$$

Splitting of the field momentum:

$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}},$$

$$\tilde{\pi}^{ij} = \partial_i V^j + \partial_j V^i - \frac{2}{d}\,\delta^{ij}\,\partial_k V^k.$$

Solving the constraint equations

The constraints yield an elliptic system for ϕ and V^i which has the structure

$$\Delta\phi = -\sum_a m_a(1+\ldots)\,\delta_a + \ldots\,,$$

$$\Delta V^i + \left(1-\frac{2}{d}\right)\partial_{ij}\,V^j = -\frac{1}{2}\sum_a (p_{ai}+\ldots)\,\delta_a + \ldots\,.$$

One can perturbatively solve this system in powers of m_a, p_{ai} and of h_{ij}^{TT} and π^{ij}_{TT} (that enter the ellipsis).

Reduced ADM Hamiltonian

If the constraint equations and the gauge conditions are satisfied, the Hamiltonian can be put into its reduced form:

$$\boxed{H_{\mathrm{r}}\big(\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \pi^{ij}_{\mathrm{TT}}\big) = -\int \mathrm{d}^d x\, \Delta\phi\big(\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \pi^{ij}_{\mathrm{TT}}\big).}$$

The equations of motion for the particles:

$$\dot{\mathbf{p}}_a = -\frac{\partial H_{\mathrm{r}}}{\partial \mathbf{x}_a}, \quad \dot{\mathbf{x}}_a = \frac{\partial H_{\mathrm{r}}}{\partial \mathbf{p}_a} \quad (a = 1, 2),$$

evolution equations for the field degrees of freedom:

$$\frac{\partial}{\partial t}\pi^{ij}_{\mathrm{TT}} = -\frac{\delta H_{\mathrm{r}}}{\delta h_{ij}^{\mathrm{TT}}}, \quad \frac{\partial}{\partial t} h_{ij}^{\mathrm{TT}} = \frac{\delta H_{\mathrm{r}}}{\delta \pi^{ij}_{\mathrm{TT}}}.$$

Further reduction of the Hamiltonian

- Legendre transformation with respect to the field variables (to get the Routhian),

$$R\big[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \dot{h}_{ij}^{\mathrm{TT}}\big] \equiv H_{\mathrm{r}} - \int \mathrm{d}^d x \, \pi^{ij}_{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}};$$

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- the field variables h_{ij}^{TT}, $\dot{h}_{ij}^{\mathrm{TT}}$ are "integrating out", i.e., replaced by their solutions as a functional of the particle variables,

$$H[\mathbf{x}_a, \mathbf{p}_a] \equiv R\big[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}(\mathbf{x}_a, \mathbf{p}_a), \dot{h}_{ij}^{\mathrm{TT}}(\mathbf{x}_a, \mathbf{p}_a)\big]$$

(where time derivatives of $\mathbf{x}_a$ and $\mathbf{p}_a$ are eliminated through the use of lower-order equations of motion).

Expansion of the functions in v/c

- PN expansion of the "instantaneous" degrees of freedom (the numbers within parentheses denote the order in v/c):

$$\phi = \phi_{(2)} + \phi_{(4)} + \dots, \qquad V^i = V^i_{(3)} + V^i_{(5)} + \dots .$$

- Time-symmetric near-zone solution for h^{TT}_{ij},

$$\Box h^{\mathrm{TT}}_{ij} = S^{\mathrm{TT}}_{ij},$$

$$h^{\mathrm{TT}}_{ij}(\mathbf{x},t) = -\frac{1}{4\pi}\int \mathrm{d}^3\mathbf{x}' \left(\frac{S_{ij}\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) + S_{ij}\left(\mathbf{x}', t + \frac{|\mathbf{x}+\mathbf{x}'|}{c}\right)}{2|\mathbf{x}-\mathbf{x}'|}\right)^{\mathrm{TT}}$$

$$= -\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{1}{(2n)!c^{2n}}\int \mathrm{d}^3\mathbf{x}' \left(|\mathbf{x}-\mathbf{x}'|^{2n-1}\right)^{\mathrm{TT}} \frac{\mathrm{d}^{2n}S}{\mathrm{d}t^{2n}}(\mathbf{x}',t)$$

$$= h^{\mathrm{TT}}_{(4)ij}(\mathbf{x},t) + h^{\mathrm{TT}}_{(6)ij}(\mathbf{x},t) + \dots .$$

UV/IR divergencies

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UV/IR divergencies

- UV divergencies—uniquely regularized by means of dimensional regularization [all the poles in $1/(d-3)$ can be removed from the Hamiltonian by adding a total time derivative].
- IR divergencies—one needs to introduce a new length scale s through a regularization factor

$$\left(\frac{|\mathbf{x}|}{s}\right)^B,$$

then take the finite part of the pole occuring for $|\mathbf{x}| \to +\infty$ at $B = 0$ in 3 dimensions (computation in d dimensions give the same result).

- Different methods of implementing of the regularization factor $(|\mathbf{x}|/s)^B$ yield the same result modulo (a time derivative and) a change in the constant C,

$$H_{4\mathrm{PN}}^{\mathrm{near\text{-}zone}\,(s)}(\mathbf{x}_a, \mathbf{p}_a; C) = H_{4\mathrm{PN}}^{\mathrm{loc}\,0}(\mathbf{x}_a, \mathbf{p}_a) + \frac{2}{5}\,\frac{G^2 M}{c^8}\big(I_{ij}^{(3)}\big)^2 \left(\ln\frac{r_{12}}{s} + C\right) + \frac{\mathrm{d}}{\mathrm{d}t}\,G[\mathbf{x}_a, \mathbf{p}_a],$$

I_{ij} denotes the (Newtonian) quadrupole moment of the binary system

$$I_{ij} := \sum_a m_a \left(x_a^i\, x_a^j - \frac{1}{3}\,\delta^{ij}\,\mathbf{x}_a^2\right).$$

4PN-accurate *conservative* 2-point-mass *matter* and *near-zone* Hamiltonian

$$
\begin{aligned}
H^{\text{near-zone}\,(s)}_{\leq 4\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a; C) = {} & (m_1 + m_2)c^2 + H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) + H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\
& + H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{3\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\
& + H^{\text{near-zone}\,(s)}_{4\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a; C).
\end{aligned}
$$

L.Blanchet and T.Damour, Phys. Rev. D **37**, 1410 (1988)

- It is impossible at the 4PN level to express (in any gauge) the near-zone metric as a functional of the instantaneous state of the source:
  4PN metric is the sum of an instantaneous functional of the source variables and of a nonlocal-in-time tail contribution.
- To compute the near-zone effect of tail-transported correlations a technique of matching between the exterior zone $r \gg r_{12}$ and the near-zone $r \ll \lambda/(2\pi)$ was employed.
- Their result depends on an arbitrary length scale, which we identify with the length scale s used in the near-zone IR regularization:
  this length scale plays the role of an intermediate scale between the scale of the system r_{12} and the wavelength $\lambda/(2\pi)$,

$$r_{12} \ll s \ll \lambda/(2\pi).$$

Tail contribution to 4PN metric

In a suitable gauge the tail contribution to 4PN metric is equal to

$$h_{00\,4\mathrm{PN}}^{\mathrm{tail}\,(s)} = h_{00\,4\mathrm{PN}}^{\mathrm{tail\,sym}\,(s)} + h_{00\,4\mathrm{PN}}^{\mathrm{rad\,reac}},$$

$$h_{00\,4\mathrm{PN}}^{\mathrm{tail\,sym}\,(s)} = -\frac{4}{5}\frac{G^2M}{c^{10}}\,x^i\,x^j\,\mathrm{Pf}_{2s/c}\left[\int_0^{+\infty}\frac{\mathrm{d}v}{v}\left(I_{ij}^{(6)}(t-v)+I_{ij}^{(6)}(t+v)\right)\right],$$

$$h_{00\,4\mathrm{PN}}^{\mathrm{rad\,reac}} = -\frac{4}{5}\frac{G^2M}{c^{10}}\,x^i\,x^j\int_0^{+\infty}\frac{\mathrm{d}v}{v}\left(I_{ij}^{(6)}(t-v)-I_{ij}^{(6)}(t+v)\right),$$

where Pf_T denotes a Hadamard partie finie with time scale T,

$$\mathrm{Pf}_T\int_0^{+\infty}\frac{\mathrm{d}v}{v}\,g(v) := \int_0^{T}\frac{\mathrm{d}v}{v}\left(g(v)-g(0)\right)+\int_T^{+\infty}\frac{\mathrm{d}v}{v}\,g(v).$$

The time-symmetric 4PN tail contribution to the action

The time-symmetric 4PN tail metric contributes to the equations of motion of the two-body system through the action

$$S_{\rm 4PN}^{\rm tail\ sym\ (\mathit{s})} = -\int \mathrm{d}t\, H_{\rm 4PN}^{\rm tail\ sym\ (\mathit{s})}(t),$$

with a nonlocal-in-time Hamiltonian

$$H_{\rm 4PN}^{\rm tail\ sym\ (\mathit{s})}(t) = -\frac{1}{5}\,\frac{G^2M}{c^8}\, I_{ij}^{(3)}(t)\,\mathrm{Pf}_{2s/c}\int_{-\infty}^{+\infty}\frac{\mathrm{d}v}{|v|}\, I_{ij}^{(3)}(t+v).$$

The time-symmetric 4PN tail contribution to the Hamiltonian

- The total Hamiltonian equals

$$H_{\rm 4PN} = H_{\rm 4PN}^{\text{near-zone}\,(s)} + H_{\rm 4PN}^{\text{tail sym}\,(s)}.$$

- The s-dependence of the both parts is

$$H_{\rm 4PN}^{\text{near-zone}\,(s)} = +\frac{2}{5}\frac{G^2M}{c^8}(I_{ij}^{(3)})^2\ln(r_{12}/s) + \dots,$$

$$H_{\rm 4PN}^{\text{tail sym}\,(s)} = +\frac{2}{5}\frac{G^2M}{c^8}(I_{ij}^{(3)})^2\ln(2s/c) + \dots,$$

therefore the dependence on s cancels between the two contributions.

- The scale s is a UV cutoff in $H_{\rm 4PN}^{\text{tail sym}\,(s)}$ and an IR one in $H_{\rm 4PN}^{\text{near-zone}\,(s)}$: this confirms the usefulness of thinking of s as being an intermediate scale between the size of the system r_{12} and the wavelength $\lambda/(2\pi)$, similar to the introduction of an intermediate scale when decomposing the calculation of the Lamb-shift in two complementary parts.

The total two-body conservative 4PN Hamiltonian

$$H_{\mathrm{4PN}}[\mathbf{x}_a, \mathbf{p}_a; C] = H_{\mathrm{4PN}}^{\mathrm{loc}}(\mathbf{x}_a, \mathbf{p}_a; C) + H_{\mathrm{4PN}}^{\mathrm{nonloc}}[\mathbf{x}_a, \mathbf{p}_a],$$

$$H_{\mathrm{4PN}}^{\mathrm{loc}}(\mathbf{x}_a, \mathbf{p}_a; C) = H_{\mathrm{4PN}}^{\mathrm{loc}\,0}(\mathbf{x}_a, \mathbf{p}_a) + \frac{2}{5}\frac{G^2 M}{c^8}\left(I_{ij}^{(3)}\right)^2 C,$$

$$H_{\mathrm{4PN}}^{\mathrm{nonloc}}[\mathbf{x}_a(t), \mathbf{p}_a(t)] = -\frac{1}{5}\frac{G^2 M}{c^8}\, I_{ij}^{(3)}(t)\, \mathrm{Pf}_{2r_{12}/c} \int_{-\infty}^{+\infty} \frac{\mathrm{d}v}{|v|}\, I_{ij}^{(3)}(t+v).$$

Poincaré invariance of the 4PN-accurate dynamics

- The 4PN-accurate dynamics, defined by

$$H^{\mathrm{loc}\,0}_{\leq 4\mathrm{PN}} = Mc^2 + H_{\mathrm{N}} + H_{1\mathrm{PN}} + H_{2\mathrm{PN}} + H_{3\mathrm{PN}} + H^{\mathrm{loc}\,0}_{4\mathrm{PN}},$$

is Poincaré-invariant in the sense of admitting ten conserved quantities whose standard Poisson brackets realize the full (PN-expanded) Poincaré algebra.

- To prove this, we have constructed the unique boost generator

$$K^i(\mathbf{x}_a, \mathbf{p}_a, t) = G^i(\mathbf{x}_a, \mathbf{p}_a) - t\,P^i(\mathbf{x}_a, \mathbf{p}_a), \quad \text{with } P^i(\mathbf{x}_a, \mathbf{p}_a) = p_{1i} + p_{2i},$$

and with a center-of-energy vector $G^i(\mathbf{x}_a, \mathbf{p}_a)$, which can be written as

$$G^i(\mathbf{x}_a, \mathbf{p}_a) = \sum_a \Big(M_a(\mathbf{x}_b, \mathbf{p}_b)\, x_a^i + N_a(\mathbf{x}_b, \mathbf{p}_b)\, p_{ai} \Big).$$

Poincaré invariance of the 4PN-accurate dynamics (cont.)

In addition, as $I_{ij}^{(3)}$ is Galileo invariant,

$$I_{ij}^{(3)} = -2\,\frac{G\mu\,M}{r_{12}^3}\left(4\,x_{12}^{\langle i}v_{12}^{j\rangle} - \frac{3}{r_{12}}\,(\mathbf{n}_{12}\cdot\mathbf{v}_{12})\,x_{12}^{\langle i}\,x_{12}^{j\rangle}\right),$$

both the C-dependent contribution to $H_{\mathrm{4PN}}^{\mathrm{loc}}$ and the nonlocal tail contribution $H_{\mathrm{4PN}}^{\mathrm{nonloc}}$ are Poincaré invariant independently of the value of the constant C.

Determination of the value of the constant C

- One needs a calculation which fully takes into account the transition between the near zone and the wave zone, without losing any information.

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- One needs a calculation which fully takes into account the transition between the near zone and the wave zone, without losing any information.
- Such a calculation was performed by D. Bini and T. Damour [Phys. Rev. D **87**, 121501(R) (2013)] in the case of the dynamics of circular orbits and in the first order in the symmetric-mass-ratio ν.
- It is enough to compare the gauge-invariant functional link $E(j;\nu)$ between the 4PN-accurate binding energy $E := H - Mc^2$ and the angular momentum $j := c\,J/(G\,m_1\,m_2)$ along circular orbits, predicted by our $H_{\rm 4PN}[\mathbf{x}_a, \mathbf{p}_a; C]$, to the corresponding link derived from the results of Bini & Damour. The comparsion yields

$$C = -\frac{1681}{1536}.$$

This result completes the determination
of the 4PN conservative dynamics of 2-point-particle systems.

4PN-accurate local Hamiltonian

$$H^{\text{local}}_{\le 4\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = (m_1 + m_2)c^2 + H_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a) + H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a)$$

$$+ H_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H^{\text{local}}_{4\text{PN}}(\mathbf{x}_a, \mathbf{p}_a)$$

(The operation "$+(1 \leftrightarrow 2)$" used below denotes the addition for each term of another term obtained by the label permutation $1 \leftrightarrow 2$.)

Newtonian/1PN Hamiltonian

$$H_{\text{N}}(\mathbf{x}_a, \mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2\, m_1} - \frac{1}{2}\, \frac{G\, m_1\, m_2}{r_{12}} + (1 \leftrightarrow 2),$$

$$c^2\, H_{1\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) = -\frac{1}{8}\, \frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\, \frac{G m_1 m_2}{r_{12}} \left(-12\, \frac{\mathbf{p}_1^2}{m_1^2} + 14\, \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)}{m_1 m_2} + 2\, \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_1 m_2} \right)$$

$$+ \frac{1}{4}\, \frac{G m_1 m_2}{r_{12}}\, \frac{G(m_1 + m_2)}{r_{12}} + (1 \leftrightarrow 2)$$

2PN Hamiltonian

$$
\begin{aligned}
c^4 H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = {} & \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1 m_2}{r_{12}}\left(5\,\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + 5\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right.\\
& \left. - 6\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right)\\
& + \frac{1}{4}\frac{G^2 m_1 m_2}{r_{12}^2}\left(m_2\left(10\,\frac{\mathbf{p}_1^2}{m_1^2} + 19\,\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\,\frac{27\,(\mathbf{p}_1\cdot\mathbf{p}_2) + 6\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)\\
& - \frac{1}{8}\frac{Gm_1 m_2}{r_{12}}\,\frac{G^2(m_1^2 + 5m_1 m_2 + m_2^2)}{r_{12}^2} + (1\leftrightarrow 2)
\end{aligned}
$$

3PN Hamiltonian

$$
\begin{aligned}
c^6\,H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = & -\frac{5}{128}\,\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\,\frac{Gm_1m_2}{r_{12}}\biggl(-14\,\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\,\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\,\mathbf{p}_1^2\,\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2} + 6\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2} \\
& - 10\,\frac{(\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2} + 24\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2} \\
& + 2\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} + \frac{(7\,\mathbf{p}_1^2\,\mathbf{p}_2^2 - 10\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3} \\
& + \frac{(\mathbf{p}_1^2\,\mathbf{p}_2^2 - 2\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3} + 15\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} \\
& - 18\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3} + 5\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\biggr) + \frac{G^2m_1m_2}{r_{12}^2}\biggl(\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4} \\
& - \frac{115}{16}\,m_1\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{1}{48}\,m_2\,\frac{25\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\,\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2} + \frac{17}{16}\,\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3} \\
& - \frac{1}{8}\,m_1\,\frac{(15\,\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11\,(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2} - \frac{3}{2}\,m_1\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}
\end{aligned}
$$

3PN Hamiltonian (cont.)

$$
+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}
$$

$$
-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\Bigg)+\frac{G^3m_1m_2}{r_{12}^3}\Bigg(-\frac{1}{48}\left(425\,m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}
$$

$$
+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(20\,m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}
$$

$$
+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Bigg)
$$

$$
+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right)+(1\leftrightarrow 2)
$$

4PN Hamiltonian

$$
\begin{aligned}
c^8\, H^{\mathrm{local}}_{4\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1\, H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\
& + \frac{G^3 m_1 m_2}{r_{12}^3}\Big(m_1^2\, H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1 m_2\, H_{442}(\mathbf{x}_a,\mathbf{p}_a)\Big) \\
& + \frac{G^4 m_1 m_2}{r_{12}^4}\Big(m_1^3\, H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2 m_2\, H_{422}(\mathbf{x}_a,\mathbf{p}_a)\Big) + \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1 \leftrightarrow 2),
\end{aligned}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{45(\mathbf{p}_1^2)^4}{128 m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2} \\
& - \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3} \\
& + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} \\
& - \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} \\
& + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3}
\end{aligned}
$$

4PN Hamiltonian (cont.)

$$
\begin{aligned}
&+\frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3}-\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^3}-\frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^3}\\
&-\frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128m_1^5m_2^3}-\frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3}\\
&-\frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^4m_2^4}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^4}-\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^4m_2^4}\\
&+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^4m_2^4}-\frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^4}+\frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^4}\\
&-\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^4}-\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^4}+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^4}\\
&+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^4m_2^4}-\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^4}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^4}\\
&-\frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^4}-\frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^4},
\end{aligned}
$$

4PN Hamiltonian (cont.)

$$
\begin{aligned}
H_{46}(\mathbf{x}_a, \mathbf{p}_a) = {} & \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} \\
& - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128m_1^5 m_2} + \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5 m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5 m_2} \\
& + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5 m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5 m_2} + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5 m_2} \\
& + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280m_1^4 m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4 m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4 m_2^2} \\
& - \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320m_1^4 m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480m_1^4 m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280m_1^4 m_2^2} \\
& - \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840m_1^4 m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4 m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4 m_2^2} \\
& + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4 m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8m_1^3 m_2^3} + \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^3 m_2^3} \\
& - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3 m_2^3} + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192m_1^3 m_2^3}
\end{aligned}
$$

4PN Hamiltonian (cont.)

$$+\frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96m_1^3m_2^3}+\frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3}-\frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3}+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3}$$
$$-\frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3}-\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4m_1^2m_2^4}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4}-\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2m_1^2m_2^4}$$
$$+\frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^2m_2^4}+\frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4}+\frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4}$$
$$-\frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4}-\frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4}+\frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4}-\frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4}+\frac{13(\mathbf{p}_2^2)^3}{8m_2^6},$$

$$H_{441}(\mathbf{x}_a,\mathbf{p}_a)=\frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384m_1^4}-\frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4}-\frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4}-\frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640m_1^3m_2}$$
$$+\frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2}+\frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2}+\frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800m_1^3m_2}$$
$$-\frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960m_1^2m_2^2}+\frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2}-\frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400m_1^2m_2^2}$$
$$+\frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400m_1^2m_2^2}+\frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2}-\frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2}+\frac{105(\mathbf{p}_2^2)^2}{32m_2^4},$$

4PN Hamiltonian (cont.)

$$
\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) = {} & \left(\frac{2749\pi^2}{8192}-\frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600}-\frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192}-\frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
& + \left(\frac{10631\pi^2}{8192}-\frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{13723\pi^2}{16384}-\frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} \\
& + \left(\frac{1411429}{19200}-\frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2} + \left(\frac{248991}{6400}-\frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2} \\
& - \left(\frac{30383}{960}+\frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{1243717}{14400}-\frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} \\
& + \left(\frac{2369}{60}+\frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} + \left(\frac{43101\pi^2}{16384}-\frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2} \\
& + \left(\frac{56955\pi^2}{16384}-\frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2},
\end{aligned}
$$

4PN Hamiltonian (cont.)

$$H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2m_1m_2},$$

$$\begin{aligned}H_{422}(\mathbf{x}_a,\mathbf{p}_a) &= \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2}\\ &+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\\ &+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2},\end{aligned}$$

$$H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2m_2^2.$$