# Gravitational turbulent instability of $AdS_5$

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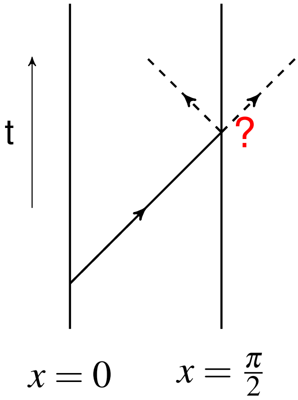
# Anti-de Sitter spacetime in $d+1$ dimensions

Manifold $\mathscr{M} = \{t \in \mathbb{R}, x \in [0, \pi/2), \omega \in S^{d-1}\}$ with metric

$$g = \frac{\ell^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x \, d\omega^2_{S^{d-1}}\right)$$

Spatial infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathscr{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$

- Null geodesics get to infinity in finite time
- AdS is **not globally hyperbolic** - to make sense of evolution one has to prescribe boundary conditions at $\mathscr{I}$
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS



# Is AdS stable?

- By the positive energy theorem AdS space is the ground state among asymptotically AdS spacetimes (much as Minkowski space is the ground state among asymptotically flat spacetimes)
- Minkowski spacetime was proved to be asymptotically stable by Christodoulou and Klainerman (1993)
- Key difference between Minkowski and AdS: the mechanism of stability of Minkowski - **dissipation of energy by dispersion** - is absent in AdS (for no-flux boundary conditions $\mathscr{I}$ acts as a mirror)
- The problem of stability of AdS has not been explored until recently; notable exceptions: proof of local well-posedness by Friedrich (1995), proof of rigidity of AdS (Anderson 2006)
- The problem seems tractable only in spherical symmetry so one needs to add matter to generate dynamics. Simple choice: a massless scalar field

# AdS gravity with a spherically symmetric scalar field

**Conjecture** (Bizoń-R 2011)

*AdS$_{d+1}$ (for $d \geq 3$) is unstable against black hole formation under arbitrarily small scalar perturbations*

Heuristic picture (supported by the nonlinear perturbation analysis and numerical evidence): due to resonant interactions between harmonics **the energy is transferred from low to high frequencies**.
The concentration of energy on finer and finer scales eventully leads to the formation of a horizon (**strongly turbulent instability**).

- The turbulent instability is absent for some perturbations, in particular there is good evidence for the existence of **stable time-periodic solutions** (Maliborski-R 2013)
- In $2+1$ dimensions there is a mass gap between AdS$_3$ and the lightest BTZ black hole. Small perturbations of AdS$_3$ remain smooth for all times but their radius of analyticity shrinks to zero as $t \to \infty$ (**weakly turbulent instability**) (Bizoń-Jałmużna 2013)

# Other models

- Due to the computational limitations the numerical analysis of stability of AdS so far has been restricted to the $1+1$ dimensional setting (spherical symmetry).
- Which features of spherical collapse in the Einstein-scalar-AdS system are model-dependent and which ones hold in general?
- Other matter models: scalar field with $m^2 < 0$, Yang-Mills (allows for different boundary conditions and admits many static solutions)
- The vacuum case seems most interesting. The analysis of weak perturbations of AdS is very similar to the scalar field case (Dias-Horowitz-Santos 2012), however *long-time* numerical simulations without a symmetry reduction appear challenging
- A partial way around: **one can evade Birkhoff's theorem in five and higher odd spacetime dimensions**

# How to bypass Birkhoff in five dimensions

- Odd-dimensional spheres admit non-round homogeneous metrics
- Homogeneous metric on $S^3$

$$g_{S^3} = e^{2B}\sigma_1^2 + e^{2C}\sigma_2^2 + e^{2D}\sigma_3^2,$$

where $\sigma_k$ are left-invariant one-forms on $SU(2)$

$$\sigma_1 + i\,\sigma_2 = e^{i\psi}(\cos\theta\, d\phi + i\,d\theta), \quad \sigma_3 = d\psi - \sin\theta\, d\phi.$$

  - $B = C = D$: round metric with $SO(4)$ symmetry
  - $B \neq C \neq D$: anisotropic metric with $SU(2)$ symmetry (squashed $S^3$)

- (Bizoń-Chmaj-Schmidt 2005): use $g_{S^3}$ as an angular part of the five dimensional metric (cohomogeneity-two triaxial Bianchi IX ansatz)

$$ds^2 = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \frac{1}{4}r^2\left(e^{2B}\sigma_1^2 + e^{2C}\sigma_2^2 + e^{-2(B+C)}\sigma_3^2\right),$$

where $A, \delta, B, C$ are functions of $(t, r)$. The biaxial case: $B = C$.

# Cohomogeneity-two biaxial Bianchi IX ansatz in AdS

$$ds^2 = \frac{\ell^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \frac{1}{4}\sin^2 x\left(e^{2B}(\sigma_1^2+\sigma_2^2)+e^{-4B}\sigma_3^2\right)\right),$$

where $A$, $\delta$, $B$ are functions of $(t,x)$. Inserting this ansatz into the vacuum Einstein equations with $\Lambda = -6/\ell^2$ we get a hyperbolic-elliptic system

$$(A^{-1}e^{\delta}\dot{B})^{\cdot} = \frac{1}{\tan^3 x}\left(\tan^3 x A e^{-\delta} B'\right)' - \frac{4e^{-\delta}}{3\sin^2 x}\left(e^{-2B}-e^{-8B}\right),$$

$$\delta' = -2\sin x\cos x\left(B'^2 + A^{-2}e^{2\delta}\dot{B}^2\right),$$

$$A' = 4\tan x\,(1-A) + A\delta' + \frac{2(4e^{-2B}-e^{-8B}-3A)}{3\tan x}.$$

- We solve this system for smooth initial data $B(0,x), \dot{B}(0,x)$ with finite mass $M = \lim_{x\to\pi/2}\sin^2 x\sec^2 x\,(1-A)$
- Asymptotic behavior near infinity ($x=\pi/2$)

$$B(t,x)\sim b_\infty(t)(\pi/2-x)^4, \quad \delta(t,x)\sim\delta_\infty(t), \quad 1-A(t,x)\sim M(\pi/2-x)^4$$

# Spectral properties

- Linearized equation:

$$\ddot{B} + LB = 0, \quad L = -\frac{1}{\tan^3 x}\,\partial_x\left(\tan^3 x\,\partial_x\right) + \frac{8}{\sin^2 x}$$

  The operator $L$ is essentially self-adjoint on $L^2([0,\pi/2), \tan^3 x\,dx)$.

- The eigenvalues and eigenfunctions of $L$ are ($k = 0, 1, \dots$)

$$\omega_k^2 = (6+2k)^2, \quad e_k(x) = d_k \sin^2 x \cos^4 x\, {}_2F_1(-k, 6+k, 4; \sin^2 x)\,,$$

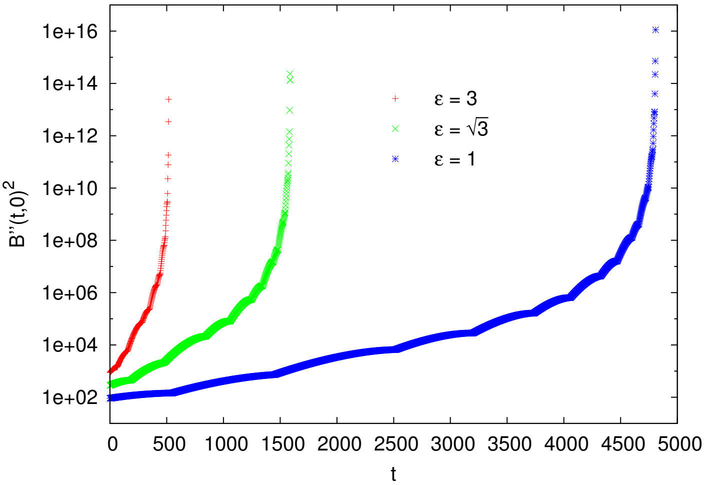
  where $d_k$ is the normalization factor ensuring that $(e_j, e_k) = \delta_{jk}$; waves are **nondispersive**: $d\omega_k/dk = \pm 2$.

- Using the generalized Fourier series $B(t,x) = \sum_k b_j(t) e_k(x)$ we express the linearized energy as the Parseval sum

$$E = \int_0^{\pi/2} \left(\dot{B}^2 + B'^2 + \frac{8}{\sin^2 x} B^2\right) \tan^3 x\,dx = \sum_k E_k\,,$$

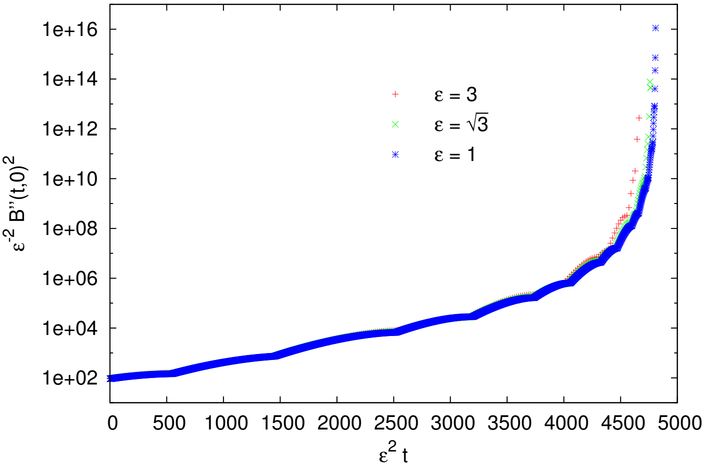
  where $E_k = \dot{b}_k^2 + \omega_k^2 b_k^2$ is the energy of the $k$-th mode.

# Blowup of the Kretschmann scalar




$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}(t,0) = 40 + 864\,B''(t,0)^2$$

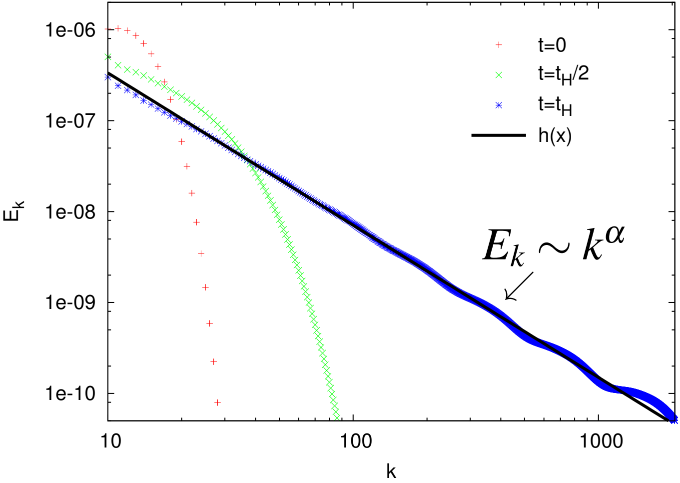
# Key evidence for instability




**Conjecture**

*$AdS_5$ is unstable against black hole formation under arbitrarily small gravitational perturbations*

# Spectrum of energy




Universal power–law exponent $\alpha \approx -1.67$ (-5/3?)

# Conclusions

- Dynamics of asymptotically AdS spacetimes is an interesting meeting point of basic problems in general relativity, PDE theory, AdS/CFT, and theory of turbulence. Understanding of these connections is at its infancy.
- Some open problems:
  - Turbulent instability is absent for some initial data. How big are these stability islands on the turbulent ocean?
  - Is the fully resonant linear spectrum necessary for the turbulent instability? (Dias, Horowitz, Marolf, Santos 2012, Maliborski-R 2014).
  - Energy cascade has the power-law spectrum $E_k \sim k^{\alpha}$ with a universal exponent $\alpha$. What determines $\alpha$?
  - What happens outside spherical symmetry? It is not clear if the natural candidate for the endstate of instability - Kerr-AdS black hole - is stable itself (Holzegel-Smulevici 2013)
  - What are the implications of all that for the AdS/CFT?

## Two kinds of stability

- Consider a nonlinear evolution equation $\frac{du}{dt} = A(u)$ and its equilibrium solution $\phi$ (that is $A(\phi) = 0$). Let $u = \phi + w$.
  The equilibrium $\phi$ is (nonlinearly) **stable** if

$$\|w(0)\|_1 \text{ is small } \Rightarrow \|w(t)\|_2 \text{ is small for all } t > 0$$

- Consider the linear equation $\frac{dv}{dt} = Lv$, where $L = A'(\phi)$.
  The equilibrium $\phi$ is **linearly stable** if

$$\|v(0)\|_1 \text{ is small } \Rightarrow \|v(t)\|_2 \text{ is small for all } t > 0$$

- Key idea of linearization: as long as $w(t)$ remains small, the nonlinear part in $A(u) = Lw + N(w)$ is negligible.
- Linear stability does not imply stability!
- The equilibrium $\phi$ is **unstable/linearly unstable** if it is not stable/linearly stable.
- In case of instability there arises a question: what happens as $t \to \infty$?