The 3+1 decomposition of Conformal Yano-Killing tensors and "momentary" charges for spin-2 field

Jacek Jezierski
Katedra Metod Matematycznych Fizyki
Uniwersytet Warszawski, ul. Hoża 69(74), 00-682 Warszawa

e-mail: Jacek.Jezierski@fuw.edu.pl

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Reports on Mathematical Physics **64** (2009) 205-221;
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How CYK tensors appear in GR?

- Geometric definition of the asymptotic flat spacetime: *strong asymptotic flatness*, which guarantees well defined total angular momentum
- Conserved quantities – asymptotic charges ($\mathscr{I}$, i^0)
- Quasilocal mass and “rotational energy” for Kerr black hole

Spacetimes possessing CYK tensor:

- Minkowski (quadratic polynomials)
- (Anti)deSitter (natural construction)
- Kerr (type D spacetime)
- Taub-NUT (new symmetric conformal Killing tensors)

Other applications:

- Symmetries of Dirac operator
- Symmetries of Maxwell equations

Conformal Yano–Killing tensors

Let $Q_{\mu\nu}$ be a skew-symmetric tensor field. Contracting the Weyl tensor $W^{\mu\nu\kappa\lambda}$ with $Q_{\mu\nu}$ we obtain a natural object which can be integrated over two-surfaces. The result does not depend on the choice of the surface if $Q_{\mu\nu}$ fulfills the following condition introduced by Penrose

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \eta_{\sigma[\lambda}Q_{\kappa]}{}^{\delta}{}_{;\delta} = 0\,, \tag{1}$$

one can rewrite equation (1) in a generalized form for n-dimensional spacetime with metric $g_{\mu\nu}$:

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \frac{3}{n-1} g_{\sigma[\lambda}Q_{\kappa]}{}^{\delta}{}_{;\delta} = 0 \tag{2}$$

or in the equivalent form:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right)\,. \tag{3}$$

Let us define

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) \tag{4}$$

Definition 1. *A skew-symmetric tensor $Q_{\mu\nu}$ is a* conformal Yano–Killing tensor *(or simply CYK tensor) for the metric g iff $\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = 0$.*

Other definitions of CYK tensors known also as *Conformal Killing forms* or *Twistor forms*:

A more abstract way with no indices of describing a CYK tensor can be found in literature: Moroianu, Semmelmann or Stepanow, where it is considered as the element of the kernel of the twistor operator

$$Q \to \mathcal{T}_{wist} Q$$

defined as follows:

$$\forall X \;\; \mathcal{T}_{wist} Q(X) := \nabla_X Q - \frac{1}{p+1} X \lrcorner \mathrm{d}Q + \frac{1}{n-p+1} g(X) \wedge \mathrm{d}^* Q \, .$$

Q is a differential p-form on n-dimensional Riemannian manifold.

However, to simplify the exposition, we prefer abstract index notation which also seems to be more popular.

The CYK tensor is a natural generalization of the Yano tensor with respect to the conformal rescalings. More precisely, for any positive scalar function $\Omega > 0$ and for a given metric $g_{\mu\nu}$ we obtain:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q, g) = \Omega^{-3}\mathcal{Q}_{\lambda\kappa\sigma}(\Omega^3 Q, \Omega^2 g)\,. \tag{5}$$

The formula (5) and the above definition of CYK tensor gives the following

Theorem 1. *If $Q_{\mu\nu}$ is a CYK tensor for the metric $g_{\mu\nu}$ than $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for the conformally rescaled metric $\Omega^2 g_{\mu\nu}$.*

It is interesting to notice, that a tensor $A_{\mu\nu}$ — a “square” of the CYK tensor $Q_{\mu\nu}$ defined as follows:

$$A_{\mu\nu} := Q_\mu{}^\lambda Q_{\lambda\nu}$$

fulfills the following equation:

$$A_{(\mu\nu;\kappa)} = g_{(\mu\nu}A_{\kappa)} \quad \text{with} \quad A_\kappa = \frac{2}{n-1}Q_\kappa{}^\lambda Q_\lambda{}^\delta{}_{;\delta} \tag{6}$$

which simply means that the symmetric tensor $A_{\mu\nu}$ is a conformal Killing tensor. This can be also described by the following

Theorem 2. *If $Q_{\mu\nu}$ is a skew-symmetric conformal Yano–Killing tensor than $A_{\mu\nu} := Q_\mu{}^\lambda Q_{\lambda\nu}$ is a symmetric conformal Killing tensor.*

Remark CYK tensor is a solution of the following conformally invariant equation ($n = \dim M = 4$):

$$\left(\Box + \frac{1}{6}\mathcal{R}\right) Q = \frac{1}{2} W(Q, \cdot)$$

$\mathcal{R} := R_{\mu\nu} g^{\mu\nu}$ – scalar curvature, $R_{\mu\nu}$ – symmetric Ricci tensor.

Moreover, if Q is a CYK tensor and the metric is Einstein then

$$K^{\mu} := Q^{\mu\lambda}{}_{;\lambda}$$

is a Killing vector field.

More precisely, one can show

$$K_{(\mu;\nu)} = \frac{n-1}{n-2} R_{\sigma(\mu} Q_{\nu)}{}^{\sigma}$$

which always implies $K^{\mu}{}_{;\mu} = 0$ and the following

Theorem 3. *If $g_{\alpha\beta}$ is an Einstein metric, i.e. $R_{\mu\nu} = \lambda g_{\mu\nu}$, then K^{μ} is a Killing vector field.*

Integrability condition

$$Q_{\lambda\kappa}{}^{;\mu}{}_{\mu} + R^{\sigma}{}_{\kappa\lambda\mu} Q^{\mu}{}_{\sigma} + Q_{\sigma\kappa} R^{\sigma}{}_{\lambda} + \frac{2}{n-1}\xi_{(\kappa;\lambda)} + \frac{1}{n-1} g_{\kappa\lambda}\xi^{\mu}{}_{;\mu} = \nabla^{\mu}\mathcal{Q}_{\mu\kappa\lambda} - \frac{n-4}{n-1}\xi_{\kappa;\lambda}\,. \tag{7}$$

For $n = \dim M = 4$ eq. (7) implies the following equation for a CYK tensor Q:

$$\nabla^{\mu}\nabla_{\mu} Q_{\lambda\kappa} = R^{\sigma}{}_{\kappa\lambda\mu} Q_{\sigma}{}^{\mu} - R_{\sigma[\kappa} Q_{\lambda]}{}^{\sigma} \tag{8}$$

It is interesting to point out that for compact four-dimensional Riemannian manifolds we have the following

Theorem 4. *Let M be a compact (without boundary) four-dimensional Riemannian manifold; then a two-form Q is a CYK tensor iff*

$$\nabla^{\mu}\nabla_{\mu} Q_{\lambda\kappa} = R^{\sigma}{}_{\kappa\lambda\mu} Q_{\sigma}{}^{\mu} + R_{\sigma[\lambda} Q_{\kappa]}{}^{\sigma}\,.$$

Proof. We need to show that equation (8) implies $\mathcal{Q}_{\lambda\kappa\mu}(Q,g)=0$.

We derive

$$\frac{2}{3}\xi_{(\mu;\lambda)}+\frac{1}{3}g_{\mu\lambda}\xi^{\nu}{}_{;\nu}-R_{\sigma(\mu}Q_{\lambda)}{}^{\sigma}+\frac{1}{2}\nabla^{\sigma}\mathcal{Q}_{\lambda\sigma\mu}=0\,,\quad 4\xi^{\mu}{}_{;\mu}+\nabla^{\sigma}\mathcal{Q}_{\nu\sigma\mu}g^{\mu\nu}=0\,,$$

which together with

$$2\xi^{\mu}{}_{;\mu}=Q^{\lambda\kappa}{}_{;\lambda\kappa}-Q^{\lambda\kappa}{}_{;\kappa\lambda}=2Q^{\kappa\sigma}R_{\sigma\kappa}=0 \tag{9}$$

and (7) gives

$$\nabla^{\mu}\nabla_{\mu}Q_{\lambda\kappa}+R^{\sigma}{}_{\kappa\lambda\mu}Q^{\mu}{}_{\sigma}+R_{\sigma[\kappa}Q_{\lambda]}{}^{\sigma}=\nabla^{\mu}\mathcal{Q}_{\mu\kappa\lambda}+\frac{1}{2}\nabla^{\sigma}\mathcal{Q}_{\kappa\sigma\lambda}\,. \tag{10}$$

Contracting the above equality with Q and assuming equation (8) we get

$$\begin{aligned} 0=\left(\nabla^{\mu}\mathcal{Q}_{\mu\kappa\lambda}+\frac{1}{2}\nabla^{\sigma}\mathcal{Q}_{\lambda\sigma\kappa}\right)Q^{\kappa\lambda} &= \nabla^{\mu}\left(\mathcal{Q}_{\mu\kappa\lambda}Q^{\kappa\lambda}\right)-\mathcal{Q}_{\mu\kappa\lambda}\nabla^{\mu}Q^{\kappa\lambda}\\ &= \nabla^{\mu}\left(\mathcal{Q}_{\mu\kappa\lambda}Q^{\kappa\lambda}\right)+\frac{1}{2}\mathcal{Q}_{\lambda\kappa\mu}\mathcal{Q}^{\lambda\kappa\mu}\,. \end{aligned} \tag{11}$$

Finally, we integrate the above formula over M, a total divergence drops out, and the integral $\int_M\sqrt{\det g}\,\mathcal{Q}_{\lambda\kappa\mu}\mathcal{Q}^{\lambda\kappa\mu}$ vanishes. This implies $\mathcal{Q}^{\lambda\kappa\mu}=0$. □

A similar result holds for a p-form Q in $2p$-dimensional M.

Let us restrict ourselves to four-dimensional manifold ($n = 4$). The Hodge-dual of $Q_{\mu\nu}$ defined as follows

$$*Q_{\kappa\lambda} = \frac{1}{2}\varepsilon_{\kappa\lambda}{}^{\mu\nu}Q_{\mu\nu}$$

gives also a two-form. Multiplying CYK equation

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}K_{\kappa} - g_{\kappa(\lambda}K_{\sigma)}\right)$$

by $\frac{1}{2}\varepsilon^{\alpha\beta\lambda\kappa}$ we get:

$$*Q_{\alpha\beta;\sigma} = \frac{2}{3}g_{\sigma[\alpha}\chi_{\beta]} + \frac{1}{3}\varepsilon_{\alpha\beta\sigma\kappa}K^{\kappa}\ , \tag{12}$$

where $\chi_\mu := *Q^{\nu}{}_{\mu;\nu}$ and $K_\mu = Q^{\nu}{}_{\mu;\nu}$. Multiplying the above equality by $\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}$, we obtain a similar formula:

$$Q_{\mu\nu;\sigma} = \frac{2}{3}g_{\sigma[\mu}K_{\nu]} - \frac{1}{3}\varepsilon_{\mu\nu\sigma\beta}\chi^{\beta}\ . \tag{13}$$

Finally, symmetrization of indices α and σ in (12) gives:

$$*Q_{\alpha\beta;\sigma} + *Q_{\sigma\beta;\alpha} = \frac{2}{3}\left(g_{\sigma\alpha}\chi_{\beta} - g_{\beta(\alpha}\chi_{\sigma)}\right)\ ,$$

which implies the following

Theorem 5. *$Q_{\mu\nu}$ is a CYK tensor iff $*Q_{\mu\nu}$ is a CYK tensor.*

In particular, it implies that Einstein manifolds possessing non-trivial CYK tensors should admit two Killing fields $K_\mu = Q^\nu{}_{\mu;\nu}$ and $\chi_\mu = *Q^\nu{}_{\mu;\nu}$. They sometimes vanish which simply means that CYK tensor or its dual is a usual Yano tensor.

For any two-form $Q_{\mu\nu}$ we have the following identity:

$$\nabla_\lambda \left(W^{\mu\lambda\alpha\beta} Q_{\alpha\beta} \right) = \frac{2}{3} W^{\mu\lambda\alpha\beta} \mathcal{Q}_{\alpha\beta\lambda} \, . \tag{14}$$

More precisely,

$$\nabla_\lambda \left(W^{\mu\lambda\alpha\beta} Q_{\alpha\beta} \right) = \nabla_\lambda \left(W^{\mu\lambda\alpha\beta} \right) Q_{\alpha\beta} + W^{\mu\lambda\alpha\beta} \nabla_\lambda \left(Q_{\alpha\beta} \right)$$

and first term vanishes for spin-2 field W, but the second one equals to right-hand side of (14) because of the symmetries of W. This implies that for any CYK tensor $Q_{\mu\nu}$ we have

$$\int_{\partial V} W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa} \mathrm{d}S_{\mu\nu} = \int_V (W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa})_{;\nu} \mathrm{d}\Sigma_\mu =$$

$$= \int_V (W^{\mu\nu\lambda\kappa}{}_{;\nu} Q_{\lambda\kappa} + W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa;\nu}) \mathrm{d}\Sigma_\mu = 0 \, .$$

The above equality implies that the flux of the quantity $W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa}$ through any two closed two-surfaces S_1 and S_2 is the same if there is a three-volume V between them (i.e. if $\partial V = S_1 \cup S_2$). We define the charge corresponding to the specific CYK tensor Q as the value of this flux.

Spin-2 field

Let us start with the standard formulation of a spin-2 field $W_{\mu\alpha\nu\beta}$ in Minkowski spacetime equipped with a flat metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. The field W can be also interpreted as a Weyl tensor for linearized gravity.

Definition 2. *The following properties:*

$$W_{\mu\alpha\nu\beta} = W_{\nu\beta\mu\alpha} = W_{[\mu\alpha][\nu\beta]},\ W_{\mu[\alpha\nu\beta]} = 0,\ \eta^{\mu\nu} W_{\mu\alpha\nu\beta} = 0 \tag{15}$$

can be used as a definition of spin-2 field W.

The $*$–operation defined as

$$({}^*W)_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu}{}_{\gamma\delta}\,, \quad (W^*)_{\alpha\beta\gamma\delta} = \frac{1}{2} W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta}$$

has the following properties:

$$({}^*W^*)_{\alpha\beta\gamma\delta} = \frac{1}{4}\varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu\rho\sigma}\varepsilon_{\rho\sigma\gamma\delta}\,, {}^*W = W^*,\ {}^*({}^*W) = {}^*W^* = -W\,,$$

where $\varepsilon_{\mu\nu\gamma\delta}$ is a Levi–Civita skew-symmetric tensor and ${}^*W$ is called dual spin-2 field.

The above formulae are also valid for general Lorentzian metrics.

Moreover, Bianchi identities play a role of field equations and we have the following

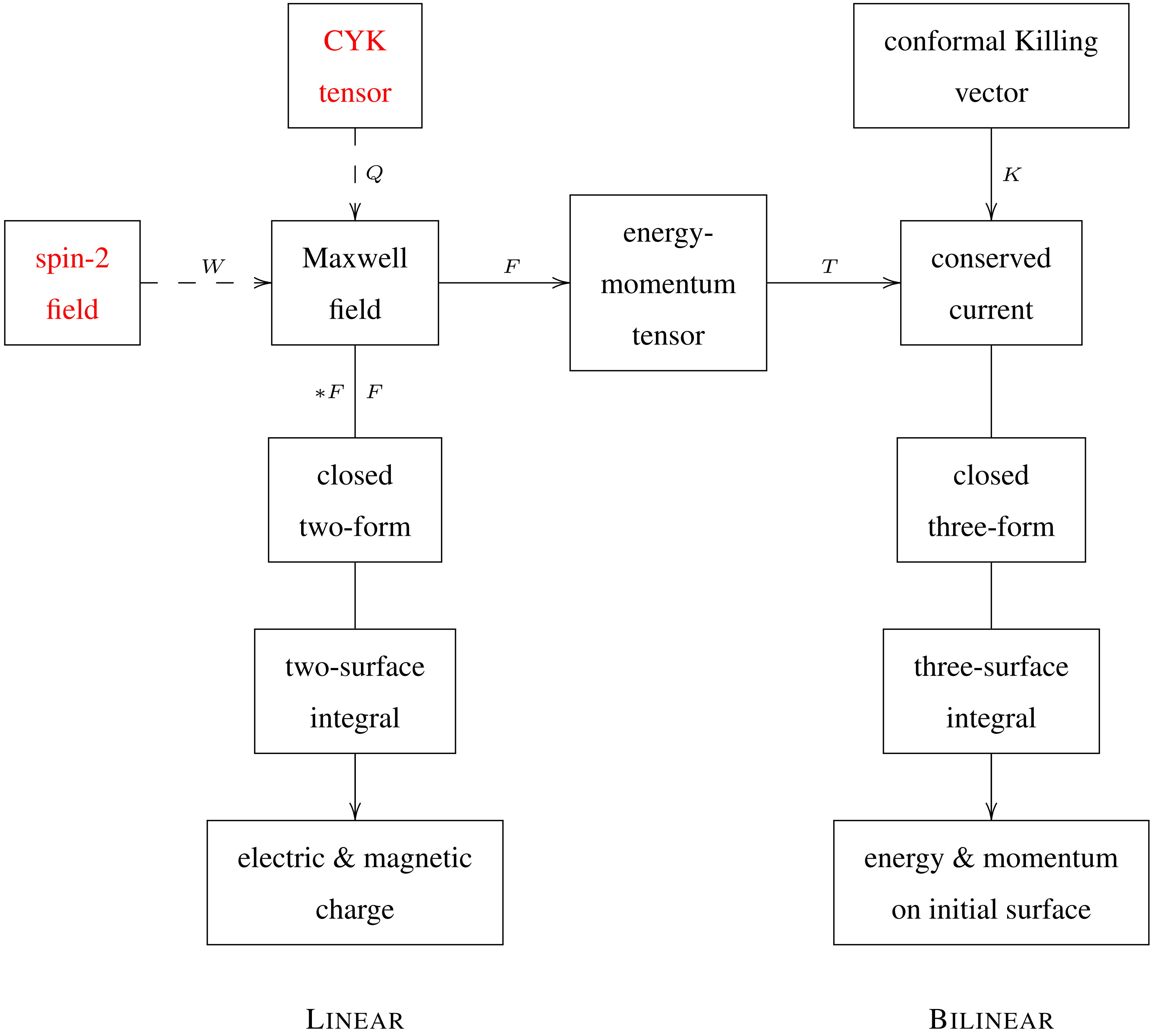
Lemma 1. *Field equations*

$$\nabla_{[\lambda} W_{\mu\nu]\alpha\beta} = 0 \tag{16}$$

are equivalent to

$$\nabla^{\mu} W_{\mu\nu\alpha\beta} = 0 \;\; or \;\; \nabla_{[\lambda} {}^{*}W_{\mu\nu]\alpha\beta} = 0 \;\; or \;\; \nabla^{\mu} \, {}^{*}W_{\mu\nu\alpha\beta} = 0 \, .$$

The equations in the above Lemma are also valid for any Ricci flat metric and its Weyl tensor.



CYK
tensor
conformal Killing
vector
Q
K
spin-2
field
W
Maxwell
field
F
energy-
momentum
tensor
T
conserved
current
*F
F
closed
two-form
closed
three-form
two-surface
integral
three-surface
integral
electric & magnetic
charge
energy & momentum
on initial surface
LINEAR
BILINEAR

Spin-2 field + CYK tensor $\longrightarrow$ Maxwell field

Let us define a skew-symmetric tensor

$$F_{\mu\nu}(W, Q) := W_{\mu\nu\lambda\kappa} Q^{\lambda\kappa}\,, \tag{17}$$

where W is the spin-2 field and Q is the CYK tensor.

Theorem 6. *For any spin-2 field W satisfying field equations (16) and any CYK tensor Q in four-dimensional spacetime the skew-symmetric tensor $F_{\mu\nu}$ (two-form F) defined by (17) fulfills vacuum Maxwell equations i.e.*

$$\mathrm{d}F = 0 = \mathrm{d}F^* \qquad \Longleftrightarrow \qquad \nabla_\lambda F^{*\mu\lambda} = 0 = \nabla_\lambda F^{\mu\lambda}\,,$$

where $F^{*\mu\lambda} = \frac{1}{2}\varepsilon^{\mu\lambda\rho\sigma}F_{\rho\sigma}$.

Proof. This is a simple consequence of the spin-2 field equations and the definition of CYK tensor. More precisely, we have

$$0 = \nabla_\lambda \left(W^{\mu\lambda}{}_{\alpha\beta} Q^{\alpha\beta} \right) = \nabla_\lambda F^{\mu\lambda}(W, Q)\,, \tag{18}$$

so half of Maxwell equations are proved. Moreover,

$$\begin{aligned} F^{*\mu\lambda}(W, Q) &= {}^*W^{\mu\lambda}{}_{\alpha\beta} Q^{\alpha\beta} = W^{*\mu\lambda}{}_{\alpha\beta} Q^{\alpha\beta} \\ &= W^{\mu\lambda}{}_{\alpha\beta} Q^{*\alpha\beta} = F^{\mu\lambda}(W, Q^*) \end{aligned} \tag{19}$$

and if Q is a CYK tensor than Q^* is also a CYK tensor hence from (18) and (19) we get the second half of Maxwell equations:

$$0 = \nabla_\lambda F^{\mu\lambda}(W, Q^*) = \nabla_\lambda F^{*\mu\lambda}(W, Q)\,. \qquad \square$$

For the special case of the *flat four-dimensional Minkowski* spacetime ($g_{\mu\nu}=\eta_{\mu\nu}$, $n=4$) the general CYK tensor assumes the following form in Cartesian coordinates (x^μ):

$$Q^{\mu\nu} = q^{\mu\nu} + 2u^{[\mu}x^{\nu]} - \varepsilon^{\mu\nu}{}_{\kappa\lambda}v^\kappa x^\lambda - \frac{1}{2}k^{\mu\nu}x_\lambda x^\lambda + 2k^{\lambda[\nu}x^{\mu]}x_\lambda \,, \tag{20}$$

where $q^{\mu\nu}$, $k^{\mu\nu}$ are constant skew-symmetric tensors and u^μ, v^μ are constant vectors.

The space of solutions for CYK equation in Minkowski spacetime is twenty dimensional. It is spanned by the following basis

$$\mathcal{T}_\mu \wedge \mathcal{T}_\nu \,, \qquad \mathcal{D}\wedge\mathcal{T}_\mu \,, \qquad *(\mathcal{D}\wedge\mathcal{T}_\mu) \,,$$
$$\mathcal{D}\wedge\mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_\mu\wedge\mathcal{T}_\nu \,,$$

where

$$\mathcal{T}_\mu = \frac{\partial}{\partial x^\mu}, \quad \mathcal{L}_{\mu\nu} = x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}, \quad \mathcal{D} = x^\mu\frac{\partial}{\partial x^\mu},$$

are generators of Poincaré group extended by scaling transformation (similarity transformations).

For any skew-symmetric tensor $t_{\lambda\kappa}$ we have defined its dual $t^*_{\mu\nu}$ as follows:

$$t^*_{\mu\nu} := \frac{1}{2}\varepsilon_{\mu\nu}{}^{\lambda\kappa} t_{\lambda\kappa} \,.$$

The construction applied to the dual spin-2 field ${}^*W$

$$\int_S {}^*W^{\mu\nu\lambda\kappa} Q_{\lambda\kappa} \mathrm{d}S_{\mu\nu} = \int_S W^{\mu\nu\lambda\kappa} Q^*{}_{\lambda\kappa} \mathrm{d}S_{\mu\nu}$$

(cf. (19)) does not give more charges because the dual tensor Q^* has the same form (20) with the following interchange:

$$q \longleftrightarrow q^* \quad k \longleftrightarrow k^* \quad u \longleftrightarrow v \,.$$

CYK equation is invariant with respect to the $*$–operation and the space of solutions is closed with respect to Hodge dual. Let us also observe that the solutions of equation (20) form a twenty-dimensional vector space. This means that

Lemma 2. *A dimension of the space of CYK 2-tensors in Minkowski spacetime is 20.*

Generally: max dimension of CYK p-forms $= \binom{n+1}{p} + \binom{n+1}{p+1}$

Remark The Theorem 1 implies that the above Lemma is also true for any conformally flat metric.

This way for each spin-2 field we can assign 20 linearly independent Maxwell fields. Each of them may carry "electromagnetic" charge which is described by

$$w_{\mu\nu} := \frac{1}{16\pi}\int_{\partial\Sigma} W(\mathcal{T}_\mu \wedge \mathcal{T}_\nu) \tag{21}$$

$$w^*_{\mu\nu} := \frac{1}{16\pi}\int_{\partial\Sigma} {}^*W(\mathcal{T}_\mu \wedge \mathcal{T}_\nu) = \frac{1}{16\pi}\int_{\partial\Sigma} W^*(\mathcal{T}_\mu \wedge \mathcal{T}_\nu)$$

$$p_\mu := \frac{1}{16\pi}\int_{\partial\Sigma} W(\mathcal{D} \wedge \mathcal{T}_\mu) \tag{22}$$

$$b_\mu := \frac{1}{16\pi}\int_{\partial\Sigma} {}^*W(\mathcal{D} \wedge \mathcal{T}_\mu) \tag{23}$$

$$j_{\mu\nu} := \frac{1}{16\pi}\int_{\partial\Sigma} W\left(\mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu\right) \tag{24}$$

$$j^*_{\mu\nu} := \frac{1}{16\pi}\int_{\partial\Sigma} {}^*W\left(\mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu\right)$$

Moreover, we can define standard energy-momentum tensor for each of them which is obviously quadratic in terms of F so it would be also quadratic in terms of W.

Natural (super-)tensor

Let us consider a tensor (proposed by Öktem 1968)

$$
\begin{aligned}
2T_{\mu\nu\alpha\beta\gamma\delta} := {} & W_{\mu\sigma\alpha\beta}W_{\nu}{}^{\sigma}{}_{\gamma\delta} + W_{\mu\sigma\gamma\delta}W_{\nu}{}^{\sigma}{}_{\alpha\beta} \\
& + W^{*}{}_{\mu\sigma\alpha\beta}W^{*}{}_{\nu}{}^{\sigma}{}_{\gamma\delta} + W^{*}{}_{\mu\sigma\gamma\delta}W^{*}{}_{\nu}{}^{\sigma}{}_{\alpha\beta}
\end{aligned} \tag{25}
$$

which is naturally related to our new conserved quantities by the following equality

$$
T^{EM}_{\mu\nu}(F(Q)) = \frac{1}{2}T_{\mu\nu\alpha\beta\gamma\delta}Q^{\alpha\beta}Q^{\gamma\delta}\,.
$$

Tensor T has the following properties:

$$
T_{\mu\nu\alpha\beta\gamma\delta} = T_{\mu\nu\gamma\delta\alpha\beta} = T_{(\mu\nu)[\alpha\beta][\gamma\delta]}\,, \quad T_{\mu\nu\alpha\beta\gamma\delta}g^{\mu\nu} = 0 \tag{26}
$$

which are simple consequences of the definition (25) and spin-2 field properties. Moreover, T is related with Bel–Robinson tensor as follows

$$
g^{\beta\delta}T_{\mu\nu\alpha\beta\gamma\delta} = T^{BR}_{\mu\nu\alpha\gamma}\,.
$$

One can also show the following properties of tensor T:

$$
\nabla^{\mu}T_{\mu\nu\alpha\beta\gamma\delta} = 0\,, \quad T_{\mu[\nu\alpha\beta]\gamma\delta} = 0\,. \tag{27}
$$

Proof. The divergence-free property for T is a consequence of spin-2 field equations which simultaneously hold for W and W^*, hence we get

$$\nabla^\mu T_{\mu\nu\alpha\beta\gamma\delta} = \frac{1}{4}\nabla_\nu \left(W^{\mu\sigma}{}_{\alpha\beta}W_{\mu\sigma\gamma\delta} + W^{*\mu\sigma}{}_{\alpha\beta}W^*{}_{\mu\sigma\gamma\delta}\right) = 0$$

where the last equality is a consequence of the following formula

$$W^{\mu\sigma}{}_{\alpha\beta}W_{\mu\sigma\gamma\delta} + W^{*\mu\sigma}{}_{\alpha\beta}W^*{}_{\mu\sigma\gamma\delta} = 0$$

which is equivalent to traceless attribute of T in (26) and can be easily checked from properties of spin-2 field with respect to $*$–operation (e.g. $*^2 = -1$). The second equality in (27) is implied by the Bianchi identity for W and W^*. $\square$

The above properties of tensor T allow us to check the following

Theorem 7. *If P, Q are CYK tensors, X is a conformal vector field and T obeys the properties (26) and (27) then*

$$\nabla^\mu \left(T_{\mu\nu\alpha\beta\gamma\delta}X^\nu P^{\alpha\beta}Q^{\gamma\delta}\right) = 0\,.$$

Gravitational charges in a 3+1 decomposition

We begin with a following simple observation.

Lemma 3. *Each CYK tensor in Minkowski spacetime can be expressed in a following way:*

$$Q = a(t)\mathcal{T}_0 \wedge X + b(t) * (\mathcal{T}_0 \wedge Y), \tag{28}$$

where X, Y are (three-dimensional) conformal Killing fields; $a(t)$, $b(t)$ are functions of time only.

One can prove this Lemma by giving the proper decomposition of the basis tensors

Let us introduce the basis of conformal Killing fields (CKV) in a flat, three-dimensional space:

$$\mathcal{T}_k := \frac{\partial}{\partial x^k}, \quad \mathcal{S} := x^k \frac{\partial}{\partial x^k}, \quad \mathcal{R}_k := \epsilon_k{}^{ij} x_i \frac{\partial}{\partial x^j}, \quad \mathcal{K}_k := x_k \mathcal{S} - \frac{1}{2} r^2 \frac{\partial}{\partial x^k} \,. \tag{29}$$

The fields written above correspond[a] (respectively) to: translation (3), scaling (1), rotation (3) and proper conformal transformation (3). Now we are able to provide the decomposition of each tensor in a form given in the Lemma 3.

$$1^\circ \quad \mathcal{T}_0 \wedge \mathcal{T}_k = \mathcal{T}_0 \wedge \mathcal{T}_k \tag{30}$$

$$2^\circ \quad \mathcal{T}_0 \wedge \mathcal{D} = \mathcal{T}_0 \wedge \mathcal{S} \tag{31}$$

$$3^\circ \quad \mathcal{T}_k \wedge \mathcal{D} = -t(\mathcal{T}_0 \wedge \mathcal{T}_k) - *(\mathcal{T}_0 \wedge \mathcal{R}_k) \tag{32}$$

$$4^\circ \quad \mathcal{D} \wedge \mathcal{L}_{0k} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_0 \wedge \mathcal{T}_k = -\frac{1}{2}t^2\mathcal{T}_0 \wedge \mathcal{T}_k - t * (\mathcal{T}_0 \wedge \mathcal{R}_k) + \mathcal{T}_0 \wedge \mathcal{K}_k \tag{33}$$

$$5^\circ \quad * (\mathcal{T}_0 \wedge \mathcal{T}_k) = *(\mathcal{T}_0 \wedge \mathcal{T}_k) \tag{34}$$

$$6^\circ \quad * (\mathcal{T}_0 \wedge \mathcal{D}) = *(\mathcal{T}_0 \wedge \mathcal{S}) \tag{35}$$

$$7^\circ \quad * (\mathcal{T}_k \wedge \mathcal{D}) = -t * (\mathcal{T}_0 \wedge \mathcal{T}_k) + (\mathcal{T}_0 \wedge \mathcal{R}_k) \tag{36}$$

$$8^\circ \quad * (\mathcal{D} \wedge \mathcal{L}_{0k} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_0 \wedge \mathcal{T}_k) = -\frac{1}{2}t^2 * (\mathcal{T}_0 \wedge \mathcal{T}_k) + t(\mathcal{T}_0 \wedge \mathcal{R}_k) + *(\mathcal{T}_0 \wedge \mathcal{K}_k) \tag{37}$$

We have obtained 20 tensors of CYK basis in a 3+1 decomposition. To calculate the charges, we have to contract each CYK tensor with a spin-2 field and then integrate the result over a two-dimensional surface.

[a] Number of vectors in each class is represented by the number in the bracket.

Finally, we can check the values of quasi-local charges for the "fully charged" solution:

$$E(\mathcal{S}) = Q(E, \mathcal{S}) = \int\limits_{S(r)} E_{ij} \mathcal{S}^j n^i dS = 8\pi m \, , \tag{38}$$

$$E(\mathcal{T}_k) = Q(E, \mathcal{T}_k) = \int\limits_{S(r)} E_{ij} \mathcal{T}_k^j n^i dS = 8\pi w_k \, , \tag{39}$$

$$E(\mathcal{R}_k) = Q(E, \mathcal{R}_k) = \int\limits_{S(r)} E_{ij} \mathcal{R}_k^j n^i dS = -8\pi d_k \, , \tag{40}$$

$$E(\mathcal{K}_k) = Q(E, \mathcal{K}_k) = \int\limits_{S(r)} E_{ij} \mathcal{K}_k^j n^i dS = 8\pi k_k \, , \tag{41}$$

$$H(\mathcal{S}) = Q(H, \mathcal{S}) = \int\limits_{S(r)} H_{ij} \mathcal{S}^j n^i dS = 8\pi b \, , \tag{42}$$

$$H(\mathcal{T}_k) = Q(H, \mathcal{T}_k) = \int\limits_{S(r)} H_{ij} \mathcal{T}_k^j n^i dS = 8\pi q_k \, , \tag{43}$$

$$H(\mathcal{R}_k) = Q(H, \mathcal{R}_k) = \int\limits_{S(r)} H_{ij} \mathcal{R}_k^j n^i dS = 8\pi p_k \, , \tag{44}$$

$$H(\mathcal{K}_k) = Q(H, \mathcal{K}_k) = \int\limits_{S(r)} H_{ij} \mathcal{K}_k^j n^i dS = 8\pi s_k \, . \tag{45}$$

CYK tensors (30) and (31) correspond to four time-independent charges:

$$\dot{Q}(E,\mathcal{T}_k) = \dot{Q}(E,\mathcal{S}) = \dot{Q}(H,\mathcal{T}_k) = \dot{Q}(H,\mathcal{S}) = 0\,. \tag{46}$$

Moreover, we obtain time-dependence of other quantities:

$$\dot{Q}(E,\mathcal{R}_k) = Q(H,\mathcal{T}_k)\,, \tag{47}$$

$$\dot{Q}(E,\mathcal{K}_k) = Q(H,\mathcal{R}_k)\,, \tag{48}$$

$$\dot{Q}(H,\mathcal{R}_k) = -Q(E,\mathcal{T}_k)\,, \tag{49}$$

$$\dot{Q}(H,\mathcal{K}_k) = -Q(E,\mathcal{R}_k)\,. \tag{50}$$

Finally, we have the following time evolution for the charges: eight quantities are constant, six of them are linear and other six are quadratic in time. More precisely, we have constant charges:

$$m(t) = m(0), \quad w_l(t) = w_l(0), \quad b(t) = b(0), \quad q_l(t) = q_l(0)\,, \tag{51}$$

linear in time:

$$p_l(t) = -tw_l(0) + p_l(0), \qquad d_l(t) = -tq_l(0) + d_l(0)\,, \tag{52}$$

and quadratic in time:

$$k_l(t) = -\frac{1}{2}t^2 w_l(0) + tp_l(0) + k_l(0)\,, \qquad s_l(t) = -\frac{1}{2}t^2 q_l(0) + td_l(0) + s_l(0)\,, \tag{53}$$

where $m(0)$ denotes initial value of m at $t=0$ and similarly for the rest of the quantities.

Electric and magnetic part of the “fully charged” spin-2 field:

$$
\begin{aligned}
E_{ij} = \quad & m\left(\frac{1}{r}\right)_{,ij} - \vec{k}\cdot\vec{\nabla}\left(\frac{1}{r}\right)_{,ij} - [(\vec{d}\times\vec{\nabla})_j\nabla_i + (\vec{d}\times\vec{\nabla})_i\nabla_j)]\frac{1}{r} + \\
& - w^l\,\frac{3}{2r^2}(\eta_{ij}n_l - n_i\eta_{jl} - n_j\eta_{il} - n_in_jn_l)\,,
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
H_{ij} = \quad & \left[\left(b - \vec{s}\cdot\vec{\nabla}\right)\nabla_i\nabla_j + (\vec{p}\times\vec{\nabla})_j\nabla_i + (\vec{p}\times\vec{\nabla})_i\nabla_j)\right]\frac{1}{r} + \\
& - q^l\,\frac{3}{2r^2}(\eta_{ij}n_l - n_i\eta_{jl} - n_j\eta_{il} - n_in_jn_l)\,.
\end{aligned} \tag{55}
$$

Let us observe that exchanging $w \to q$, $k \to s$, $d \to -p$ and $m \to b$ in the electric part E we get magnetic part H. $E(w \to q, k \to s, d \to -p, m \to b) = H$ represents spin-2 field version of electromagnetic symmetry between electric and magnetic monopole.

Traditional Poincaré charges Let us observe that traditional relations between angular momentum (or center of mass) and Killing vectors (e.g. ADM or Komar formula) are substituted by conformal acceleration. More precisely, we have the following table:

KV	Charges	CKV		
$\mathcal{T}_0$	$p_0 \leftrightarrow m$	$\mathcal{S}$	(1)	energy
$\mathcal{T}_k$	$p_k \leftrightarrow \mathbf{p}$	$\mathcal{R}_k$	(3)	linear momentum
$\mathcal{L}_{kl}$	$j_{kl} \leftrightarrow \mathbf{s}$	$\mathcal{K}_k$	(3)	angular momentum
$\mathcal{L}_{0k}$	$j_{0k} \leftrightarrow \mathbf{k}$	$\mathcal{K}_k$	(3)	center of mass

Other quantities: $\mathbf{b}$ – dual mass, $\mathbf{d}$ – dual momentum , $\mathbf{w}$ – linear acceleration, $\mathbf{q}$ – angular acceleration are usually vanishing, if we want to have global "potentials" (linearized metric h like vector potential A for magnetic monopole). However, some parameters in Einstein metrics can be interpreted as topological charges, e.g. dual mass appears in Taub-NUT solution and dual momentum in Demiański metrics. In *J.B. Griffiths and J. Podolsky, A new look at the Plebański–Demiański family of solutions* a large class of metric tensors is given. It would be nice to check, if some parameters in those spacetimes correspond to charges $\mathbf{q}$ and $\mathbf{w}$ in some asymptotic regime.

Let's begin with writing down the Schwarzschild metric in the parametrization (t, r, θ, ϕ):

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{56}$$

Now let us introduce a new coordinate $\bar{r}$ defined by the equality $r = \bar{r}(1 + \frac{M}{2\bar{r}})^2$. Schwarzschild metric in coordinates (t, $\bar{r}$, θ, ϕ) takes the form:

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(\frac{1 - M/2\bar{r}}{1 + M/2\bar{r}}\right)^2 dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4 \left[d\bar{r}^{\,2} + \bar{r}^{\,2}\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]. \tag{57}$$

We observe that for a fixed value of t, the 3-D metric is conformally flat (i.e. it takes the form of a flat-space metric multiplied by a conformal factor $\left(1 + \frac{M}{2\bar{r}}\right)^4$). Using this fact we conclude that we have a set of ten conformal Killing fields, which are identical to the ones that we had for a flat 3-D space, since both metrics are conformally equivalent. Moreover, using the formula $K_{ik} = \frac{1}{2N}(N_{i|k} + N_{k|i} - \frac{\partial g_{ik}}{\partial t})$ we can easily check that the extrinsic curvature tensor K_{ik} vanishes.

Definition of momentary charges Previously we have used the contraction of a spin-2 field with CYK tensors to define global charges, then we have shown that CYK tensors can be expressed as contractions of electric (or magnetic) part with conformal Killing fields. Following this lead we will try to provide a definition of the momentary charges for the Schwarzschild metric as contractions of E and H with conformal fields (ignoring the fact that we have not enough CYK tensors for this metric).

In the future we would like to apply this construction for the case of asymptotically flat initial data. It is well known that some spacetimes admit (exact) CYK tensors but in general one should consider asymptotic CYK tensors which correspond to the notion of *strong asymptotic flatness*. The existence of asymptotic conformal Killing vectors is less restrictive and it should lead to the definition of global momentary charge for different asymptotics at spatial infinity. In particular, angular momentum and center of mass correspond to conformal acceleration $\mathcal{K}$.

The content of this talk is the answer to the following question:
What is the analog of Coulomb solution (electric and magnetic monopole) for the spin-2 field?
For spin-1 field the solution is "monopole", for spin-2 field we have also dipole part. In Maxwell theory we have only time independent charges, for gravity we get also time-dependent quantities. The "wave part" of the theory starts from dipoles ($l = 1$) for electrodynamics and respectively quadrupoles ($l = 2$) for gravity. Hence the "charged part" for spin-1 field is represented by $l = 0$ but for spin-2 field we have $l = 0$ and $l = 1$. Finally, the analog of the electric-magnetic monopole in electrodynamics is given by the mono-dipole solution for spin-2 field.

Let us consider a three-slice Σ equipped with a three-metric γ_{ij} which is asymptotically flat:

$$\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 (\eta_{ij} + h_{ij}) \ , \tag{58}$$

where η_{ij} is flat and h_{ij} is a “small perturbation”

Theorem 8.

$$\gamma_{ij} = \left(1 + \frac{M}{2r}\right)^4 (\eta_{ij} + h_{ij}) \ , \tag{59}$$

we assume the following asymptotics when $r \to \infty$:

$$h_{ij} = O\left(r^{-1-\epsilon}\right) \ , \tag{60}$$

$$(\Gamma_{\eta+h})^a{}_{ij} = O\left(r^{-2-\epsilon}\right) \ , \tag{61}$$

$$K_{ij} = O\left(r^{-3}\right) \ , \tag{62}$$

$$E_{ij} = O\left(r^{-3}\right) \ , \tag{63}$$

$$B_{ij} = O\left(r^{-3}\right) \ , \tag{64}$$

where $\epsilon > 0$.

Then there exists finite limit at $r \to \infty$ *for the asymptotic charges corresponding to the mass, center of mass, linear and angular momentum.*

Schwarzschild – de Sitter in Painleve – Gullstrand form:

$$ds^2 = -\left(1 - \frac{2M}{r} - br^2\right) dt_p^2 - 2\sqrt{\frac{2M}{r} + br^2}\, dr dt_p + dr^2 + r^2\Omega^2 \,. \tag{65}$$

here $b = -\frac{\Lambda}{3}$ corresponds to cosmological constant.

Initial data:

lapse N and shift N^i:

$$N = 1\,, \tag{66}$$

$$N_r = -\sqrt{\frac{2M}{r} + br^2}\,, \qquad N_\theta = 0\,, \qquad N_\phi = 0\,. \tag{67}$$

extrinsic curvature:

$$K_{ij} = \frac{1}{2N}\left(N_{i|j} + N_{j|i} - \partial_{t_s} g_{ij}\right) \,. \tag{68}$$

has the following non-vanishing components:

$$K_{rr} = \frac{M - br^3}{\sqrt{2Mr^3 + br^6}}\,, \tag{69}$$

$$K_{\theta\theta} = -\sqrt{2Mr + br^4}\,, \tag{70}$$

$$K_{\phi\phi} = -\sqrt{2Mr + br^4}\,\sin^2\theta\,. \tag{71}$$

with the trace:

$$K = K_{ij}\gamma^{ij} = -\frac{3(M + br^3)}{\sqrt{2Mr^3 + br^6}}\,. \tag{72}$$

Moreover, electric part

$$E_{ij} = -KK_{ij} + K_{ik}K^k{}_j + \frac{2}{3}\Lambda\gamma_{ij} = -KK_{ij} + K_{ik}K^k{}_j + 2b\gamma_{ij} \,. \tag{73}$$

posseses the following non-vanishing components:

$$E_{rr} = \frac{2M}{r^3} \,, \tag{74}$$

$$E_{\theta\theta} = -\frac{M}{r} \,, \tag{75}$$

$$E_{\phi\phi} = -\frac{M}{r} \sin^2\theta \,. \tag{76}$$

and magnetic part vanishes.

Lesson: in the case of the foliation with flat leaves (Painleve–Gullstrand foliation) the extrinsic curvature behaves asymptotically like $r^{-3/2}$ which is too weak from ADM point of view. Nevertheless "electromagnetic mass" is well defined and is quasi-local. (it does not depend on the choice of the sphere and cosmological constant $b = -\frac{\Lambda}{3}$)